1637 devised a somewhat complicated construction in which an hyperbola was employed. In 1905 F. Gomese Teixeira indicated a method which has the advantage of requiring only the use of straight lines and circles. However, the simplest and most natural method is the classic one of constructing the curve of intersection of a plane with a surface of revolution of vertical axis. When the method of double projection is used to represent the surfaces in question and the second plane of projection is taken to be parallel to the cutting plane,[1] then the vertical projection of the curve becomes identical with the objective curve, and the end sought is attained by a solution the details of which can be found in a letter which I addressed to M. Teixeira and which he published in Vol. IX (pp. 193–196) of the *Annaes scientificos da Academia polytechnica do Porto*. The thing is so simple and natural that I could never persuade myself to make it the subject of a publication as such. However, it appears that no one had thought of it before I did. If it did not concern itself with such a little thing, one might see in all this a confirmation of the observation of Jacobi that discoveries are born of the contact of theories which were at first separate.

Permit me to return to the construction of the ovals of Descartes discovered by Quetelet. These curves, as you know, possess three real foci which are situated on a straight line. But one can consider curves entirely analogous to these of which one focus alone is real while the other two are conjugate imaginaries. A. Cayley was the first to consider these curves in his "Note on the cartesian with two imaginary axial foci." [2] He called them Cartesian lines and established some of their properties, but he did not arrive at a simple construction nor did he determine their form. Now all of these things can be obtained by a consideration of geometry of three dimensions because there exists a simple relation between these curves and the ovals of Descartes, which have already been considered in a preceding paragraph.

A cartesian curve may be obtained as the orthographic projection of the curve of intersection of two cones of revolution which are conjugate imaginaries and of which the axes aıe parallel to each other, provided the plane of projection is normal to the axes. As descriptive geometry concerns itself exclusively with real elements, this remark does not lead at once to the desired result; but it is natural to ask whether it is not possible to substitute for the two imaginary cones two real surfaces of the pencil of surfaces which they determine. Now I have recently demonstrated in the note "Fasci di quàdrichi rotonde e Curve cartesiane," [3] of which I have the honor to send you a copy, that "through the intersection of two real or conjugate imaginary cones of revolution of which the axes are parallel, there passes a third cone of revolution of which the axis is parallel to the axes of the given cones and also a sphere, both of these surfaces being real; if the two given cones are real, the intersection is composed of two branches and its projection on the plane perpendicular to the axes is a pair of conjugate ovals of

[1] See my *Vorlesungen*, Vol. II, p. 258.

[2] *Proc. of the London Math. Society*, Vol. III, 1871–72 or also *The Collected Papers*, Vol. VII, pp. 241–243.

[3] *Rend. della R. Accademia dei Lincei*, sèance du 17 Mars, 1918.

Descartes; but if the given cones are conjugate imaginaries, the intersection has only one branch and then its projection is a cartesian line." It is thus demonstrated that the ovals of Descartes and the cartesian curves may be obtained in the same manner, namely, as the orthographic projection of the curve of intersection of a real sphere by a real cone of revolution, and that the first or second type of curve is obtained according as the cone actually penetrates or merely "bites" the sphere. Now since a sphere and a cone of revolution may be regarded as two surfaces of revolution with parallel axes, the orthographic projection of their intersection may be obtained by the aid of a classic construction of descriptive geometry to which I have already had occasion to refer. In Fig. 1 I have carried through this construction, supposing, as is permissible, that the vertical plane of projection is parallel to the plane determined by the axis of the cone and the center of the sphere. In the figure the horizontal projection only of the curve of intersection is constructed, and thus I have arrived at the *first construction which I know of cartesian lines.*

I will finish with the remark that that which precedes enables us to complete another theorem of Quetelet. In effect, this geometer demonstrated that if one has a sphere and a right cone and makes a stereographic projection of the curve of penetration of these surfaces, the eye being placed at an extremity of the diameter of the sphere which is parallel to the axis of the cone and the plane of projection taken perpendicular to this axis, then the projection will be a pair of conjugate ovals of Descartes. Now, if the cone instead of penetrating the sphere, merely "bites" it, the projection will be a cartesian line.

## II.

The application of considerations of geometry of space to geometry of the plane is one of the most brilliant ideas of Monge. But it is remarkable that in the hands of some of his pupils (I cite particularly Hachette and Olivier) it has undergone a strange and vexatious deviation to which I wish to call your attention.

One of the principal aims of descriptive geometry is to replace by drawings in a plane the purely ideal constructions of geometry of three dimensions. Until some physicist will have taught us how to *design in space*, the inverse process, namely that of replacing a planometric construction by a stereometric one, represents from the graphic point of view a veritable step backward. Nevertheless, in several works of the two geometers whom I have mentioned, one finds space constructions which have for their object the determination of tangents to particular curves—constructions which give the deplorable delusion of having solved problems which, on the contrary, are yet unsolved. I think, on the contrary, that descriptive geometry should have as a basis a complete and detailed knowledge of all of plane geometry. It is for this reason that I long for the publication of a comprehensive work of which Part I shall comprise all constructive geometry of plane figures, while Part II shall be a treatise on descriptive geometry. In order to render more striking the utility of such a fusion, I will cite an example.

In 1826, H. Garbinski, the apostle of descriptive geometry in Poland, published in the *Annales de Mathématiques* de Gergonne (T. XVI, pp. 167–172), an article on the conical spiral, a curve which he conceived in generalizing the definition of the ordinary cylindrical helix, but which B. Pascal had considered

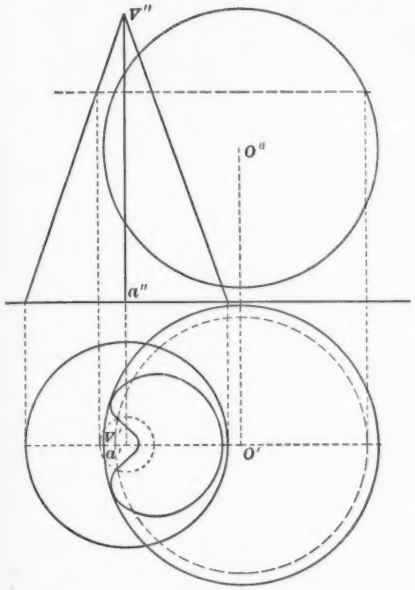


FIG. 1.

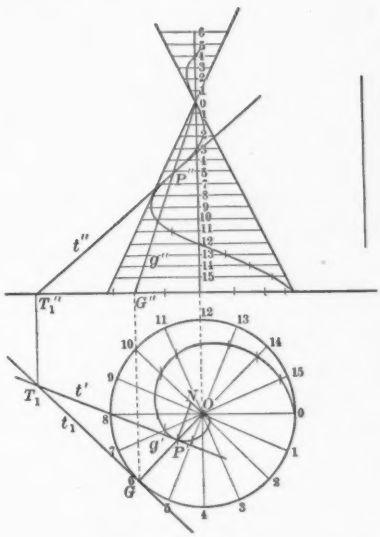


FIG. 2.

two centuries before. This curve is the trajectory described by a point, which, starting at the vertex of a right circular cone, describes with uniform motion a generatrix, which in turn moves around the surface of the cone with a constant velocity. From this definition the following properties may be deduced: (1) The orthographic projection of the conical spiral on a plane normal to the axis of the cone is a spiral of Archimedes. (2) The conical spiral lies on the surface of a right helicoid, which is coaxial with the cone of this spiral. In order to represent the curve in question by the method of Monge, one can use a process which is entirely analogous to that which is used to represent the cylindrical helix (see Fig. 2). The curve may be represented analytically by the following equations:

$$(1) \qquad x = \frac{r\omega}{2\pi}\cos\omega, \qquad y = \frac{r\omega}{2\pi}\sin\omega, \qquad z = l - \frac{l\omega}{2\pi},$$

in which $l$ is the height of the cone, $r$ is the radius of its base, and $\omega$ is a parameter. These equations prove that the orthographic projection $\Gamma'$ of the spiral $\Gamma$ on the plane $xy$ is the spiral of Archimedes

$$(2) \qquad \rho = \frac{r\omega}{2\pi}$$

and that $\Gamma$ lies upon the surface of the right helicoid

$$\frac{y}{x} + \tan\frac{2\pi z}{l} = 0. \tag{3}$$

Equation (2), on differentiation, yields the equation

$$\frac{d\rho}{d\omega} = \frac{r}{2\pi}, \tag{4}$$

and consequently the curve $\Gamma'$ enjoys the remarkable property that its polar subnormal $ON$ in any point whatever $P'$ has the constant value $r/2\pi$. It is then, indeed, easy to construct the normal, and consequently the tangent $t'$ to $\Gamma'$ at the arbitrary point $P'$, that is to say, the horizontal projection of the tangent $t$ at a general point of the conical spiral. To complete the representation of the tangent $t$, let us remark (1) that $t''$ passes through $P''$, (2) that $t$ lies in the plane $\tau$ which is tangent to the given cone along the generatrix $g$ which passes through the point $P$. Now if $G$ is the horizontal trace of $g$, the horizontal trace of the plane $\tau$ is the tangent $t_1$ at the point $G$ to the circular base of the given cone. Then the intersection $T_1$ of $t'$ and $t_1$ is the horizontal trace of the required tangent, while $t''$ is the line which joins $T_1''$ and $P''$.

Now in place of having recourse to this simple solution, Garbinski, whose knowledge of the spiral of Archimedes seemed rather limited, made use of the fact that the curve $\Gamma$ lies on the helicoid (3), and therefore that the tangent sought lies in the corresponding tangent plane to this surface. He thus constructed this plane by the aid of a related hyperbolic paraboloid. From another point of view this last artifice was not indispensable, for as Vallès remarked at once,[1] it is most simple to make use of one of the cylindrical helices lying on the helicoid. As a consequence of all of these considerations, Garbinski and Vallès gave constructions for the tangent to the spiral of Archimedes which were extremely more complicated than the construction which is a result of the constancy of the subnormal.

I wish to add that the preceding construction of the tangent to the conical spiral may serve also an analogous end for the *conical helix* (that is the curve which is now called, on account of one of its remarkable properties, the helical cylindrical-conic). This curve has the property of projecting itself orthographically on the plane of the base of the cone into a logarithmic spiral. Now the tangents of this curve make constant angles with the corresponding radii vectores. Consequently nothing is easier than to construct the horizontal projection of the tangent to the conical helix. Then the vertical projection is obtained by the aid of the tangent plane to the given cone precisely as in the case of the conical spiral.

## III.

Although descriptive geometry may have been conceived by Monge as a practical science, I have not been able to find in the publications of this great

[1] *Annales de Mathématiques*, T. XVI, pp. 376–377.

geometer any word relative to the necessity for obtaining the *most simple solutions possible*, that is to say, those that are at the same time the most economical and the most exact. The criterion of simplicity for the choice among several solutions of the same problem is entirely modern; to be sure it has not yet been applied to its full extent, with the result that researches in this direction make it possible, if I do not deceive myself, to penetrate farther into several domains which are now regarded by some writers as having been exhaustively studied. To prove to you that I do not allow myself to be deceived, I am going to return to a classic problem, namely, to the construction of the tangent $t$ at a general point $P$ (Fig. 4) of the cylindrical helix, ordinarily represented by the method of Monge. It is known[1] that to effect this construction one begins by drawing the horizontal projection$t'$ of the tangent $t$ (that is, the tangent $t'$ at the point $P'$ of the circular base of the given cylinder). After determining the horizontal trace $T_1$ of $t$, the second projection $t''$ of $t$ is found by connecting $T_1''$ with $P''$. But in order to determine $T_1$, it is necessary to find a rectilinear segment equal in length to the arc of the circular base which is comprised between $P'$ and the point where the helix pierces the horizontal plane. Now the rectification of a circular arc is a very delicate transcendental operation which may be performed by a simple approximation process,[2] and which must be repeated for each point of the curve where the tangent is required. Consequently I flatter myself that I may interest you in a process which holds for all points of the helix—a unique transcendental operation which is more simple than the rectification of the circular arc. In order to establish this result, I will make use of the following remark (to which, moreover, we will have recourse in several analogous questions): "If one considers two projections $\Gamma'$ and $\Gamma_0$ (parallel or central) of a curve $\Gamma$ on the same plane $\sigma$, the tangents to these curves at any two corresponding points $P'$ and $P_0$ will intersect in the trace on the plane $\sigma$, of the tangent at the point $P$, to the objective curve $\Gamma$." To apply it to our question, let us recall[3] that if one projects the helix, represented by the equations

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z = \frac{H}{2\pi}\varphi, \tag{5}$$

from the point $(0, 0, c)$ on the plane $xy$, there results the hyperbolic spiral having for its polar equation

$$\rho\left(\frac{2\pi c}{H} - \omega\right) = \frac{2\pi c r}{H}, \tag{6}$$

from which by differentiation one obtains the equation,

$$\frac{1}{\rho}\frac{d\rho}{d\omega}\left(\frac{2\pi c}{H} - \omega\right) = 1. \tag{7}$$

On dividing member by member equation (6) by equation (7), one obtains the

[1] See *Vorlesungen*, Vol. II, p. 137.
[2] See *Vorlesungen*, Vol. II, p. 63.
[3] *Vorlesungen*, Vol. II, pp. 130–32.

relation

$$\rho^2 \frac{d\omega}{d\rho} = \frac{2\pi cr}{H} \tag{8}$$

which shows that the curve (6) possesses the property that its polar subtangent has a constant value $l = 2\pi cr/H$. In order to determine a segment equal in

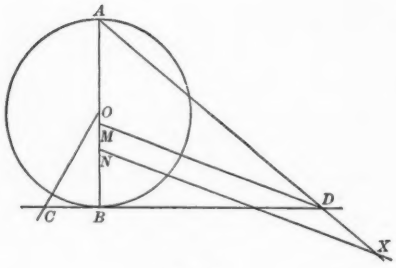


FIG. 3.

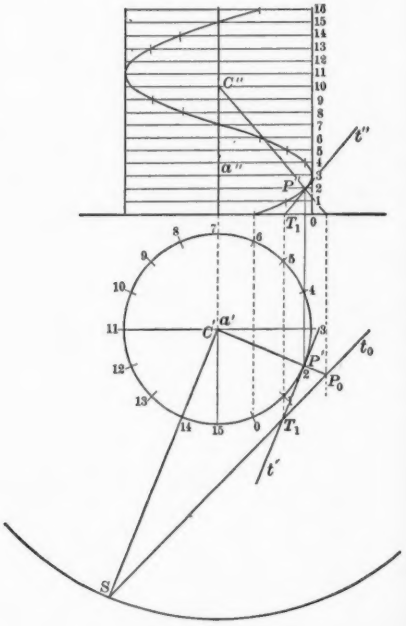


FIG. 4.

length to $l$ (Fig. 3), let us construct by the method of Kochanski,[1] or by any other process, a line $AD$ equal to the half circumference $\pi r$ of the circle of diameter $AB = 2r$. Let us now lay off on $AB$ the segments $AM = H/2$ and $AN = c$, and then join $M$ to $D$ and draw $NX$ parallel to $MD$. Then

$$\frac{AX}{AN} = \frac{AD}{AM}, \quad \text{or indeed} \quad \frac{AX}{c} = \frac{\pi r}{H/2}, \quad \text{whence} \quad AX = \frac{2\pi cr}{H} = l.$$

We may now pass to Fig. 4, which represents the given helix. Let $P \equiv (P', P'')$ be any point of the helix and $P_0$ its projection from the center $C \equiv (C', C'')$ on the horizontal plane. Describe the circle of center $a'$ and radius $l$ and draw in this circle the radius $a'S$ making with the line $a'P'$ an angle of 90° in the negative sense of rotation. Then the line $SP_0$ will be the tangent at the point $P_0$ to the hyperbolic spiral, that is to say, the central projection, from the center

[1] *Vorlesungen*, Vol. II, p. 63.

$C$, of the tangent, at the point $P$, to the given helix. This tangent cuts the tangent $t'$, which is drawn at the point $P'$ to the circle $\Gamma$, in the horizontal trace $T_1$ of the required tangent. Etc.

GENOA, May–June, 1918.

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## DIFFERENCE QUOTIENTS.

By J. P. BALLANTINE, Cambridge, Mass.

The whole subject of interpolation, in the case of equally spaced ordinates, by means of algebraic polynomials is commonly based on the subject of finite differences. The method, however, fails for unequal spacing. The purpose of this paper is to show, that by a suitable extension of the conception of the difference quotient, the method can be generalized for the case when the ordinates are spaced according to any law.[1]

Let $f(x)$ be any single valued function. Plot the curve $y = f(x)$. Pick out on the curve the $(n+1)$ points whose $x$ coördinates are $1, 2, \cdots, (n+1)$, respectively. By means of the method of finite differences, find the equation of the curve of the form $y = P_n(x)$ which passes through the $(n+1)$ points. For convenience, call this curve a secant of degree $n$, noticing that a secant of the first degree is an ordinary secant. It is easily verified that the $n$th derivative of this secant is precisely equal to the difference of order $n$ of the ordinates of the $(n+1)$ points.

Now let us take our $(n+1)$ points on the curve at random. We have to define the difference quotient of order $n$ at these $(n+1)$ points in such a way that when the values of $x_1, x_2, \cdots, x_{n+1}$ are $1, 2, \cdots, n+1$ respectively, it will reduce to a difference of order $n$.

We can define the secant through these $(n+1)$ points as the curve of the form $y = P_n(x)$ which passes through them. By way of a more explicit notation, $y = P_n(x;\ 1,\ n+1)$ will be the equation of the $n$th degree secant at the points $(x_1, y_1), (x_2, y_2) \cdots (x_{n+1}, y_{n+1})$. It follows that

$$P_n(x_i;\ 1,\ n+1) = y_i, \qquad 1 \leq i \leq n+1.$$

DEFINITION. The difference quotient of order $n$, taken at the $(n+1)$ points $(x_1, y_1), (x_2, y_2) \cdots (x_{n+1}, y_{n+1})$ is the $n$th derivative of the $n$th degree secant through those points. It is denoted thus $Q^n_{1,\,n+1}$:

$$Q^n_{1,\,n+1} = \frac{d^n}{dx^n} P_n(x;\ 1,\ n+1).$$

Consider the special polynomial and associated equation

$$P_{n+1}(x;\ 1,\ n+2) - P_n(x;\ 1,\ n+1),$$

(1) $$P_{n+1}(x_i;\ 1,\ n+2) - P_n(x_i;\ 1,\ n+1) = y_i - y_i = 0,\ 1 \leq i \leq n+1.$$

[1] Cf. A. A. Markoff, *Differenzenrechnung*, 1896, p. 10.

From the factor theorem for polynomials, we may say that:

$$(2)\quad P_{n+1}(x;\ 1, n+2) - P_n(x;\ 1, n+1) = K(x-x_1)(x-x_2)\cdots(x-x_{n+1}).$$

Now, determine $K$ by $(n+1)$ differentiations:

$$(3)\qquad Q^{n+1}_{1,\,n+2} = K(n+1)!.$$

Using this value of $K$ in (2), and transposing, we have

$$(\mathrm{I})\quad P_{n+1}(x;\ 1, n+2) = P_n(x;\ 1, n+1) + \frac{Q^{n+1}_{1,\,n+2}}{(n+1)!}(x-x_1)(x-x_2)\cdots(x-x_{n+1}).$$

In this way a secant of degree $(n+1)$ is expressed in terms of one of degree $n$ and a difference quotient. We know how to write the equation of a secant of degree one or zero, and so our problem of writing the equation of a secant of the $n$th degree at $(n+1)$ arbitrary points is solved, except for determining the difference quotient.

In the same way as we inferred the truth of (2), we may show that

$$(2')\quad P_{n+1}(x;\ 1, n+2) - P_n(x;\ 2, n+2) = K(x-x_2)(x-x_3)\cdots(x-x_{n+2}).$$

It is not obvious that the two arbitrary constants must be the same, until one notices that in each case $K$ must be equal to the coefficient of $x^{n+1}$ in the polynomial, $P_{n+1}(x;\ 1, n+2)$.

Subtracting (2′) from (2), we have

$$(4)\quad K(x_{n+2}-x_1)(x-x_2)(x-x_3)\cdots(x-x_{n+1}) = P_n(x;\ 2, n+2) - P_n(x;\ 1, n+1).$$

Differentiating $n$ times, we obtain

$$(5)\qquad K(x_{n+2}-x_1)(n!) = Q^n_{2,\,n+2} - Q^n_{1,\,n+1}.$$

Then by means of the value of $Q^{n+1}_{1,\,n+2}$ as obtained in (3), we have

$$(\mathrm{II})\qquad Q^{n+1}_{1,\,n+2} = \frac{Q^n_{2,\,n+2} - Q^n_{1,\,n+1}}{\dfrac{x_{n+2}-x_1}{n+1}}.$$

We are now in a position to form a table of difference quotients. Formula (II) tells us, that besides forming a difference, we must divide by a denominator, which is simply the average of the $(n+1)$ intervals of $x$ between the $(n+2)$ points at which $Q^{n+1}_{1,\,n+2}$, the entry we are at the time computing, is taken. It may be denoted thus:

$$(6)\qquad \Delta_k x_i = \frac{x_{k+i} - x_i}{k}.$$

Applying the definition of the difference quotient of order zero, we notice that the secant of order zero through the point $(x_i y_i)$ is given by the equation: $y = y_i$. Without differentiation, we find: $Q^0_i = y_i$.

The table of difference quotients may be arranged as follows:

$$\begin{array}{cccc|cccccc} & & & x_1 & y_1 = Q_1^0 & & & & & \\ & & \Delta_1 x_1 & & & \Delta Q_{1,2}^0 & Q_{1,2}^1 & & & \\ & \Delta_2 x_1 & & x_2 & y_2 = Q_2^0 & & & \Delta Q_{1,3}^1 & Q_{1,3}^2 & \\ \Delta_3 x_1 & & \Delta_1 x_2 & & & \Delta Q_{2,3}^0 & Q_{2,3}^1 & & & \Delta Q_{1,4}^2 \quad Q_{1,4}^3 \\ & \Delta_2 x_2 & & x_3 & y_3 = Q_3^0 & & & \Delta Q_{2,4}^1 & Q_{2,4}^2 & \\ & & \Delta_1 x_3 & & & \Delta Q_{3,4}^0 & Q_{3,4}^1 & & & \\ & & & x_4 & y_4 = Q_4^0 & & & & & \end{array}$$

The elements $x_i$ and $y_i$ are assigned.

All the elements to the left of these can be computed by relation (6) above, and those on the right by the following two relations:

$$\Delta Q_{i,\,i+k+1}^k = Q_{i+1,\,i+k+1}^k - Q_{i,\,i+k}^k, \tag{7}$$

$$Q_{i,\,i+k}^{k+1} = \frac{\Delta Q_{i,\,i+k+1}^k}{\Delta_{k+1} x_i}. \tag{8}$$

One way of writing the equation of the secant through the four points $(x_1y_1)$, $(x_2y_2)\cdots$ would be:

$$y = y_1 + \frac{Q_{1,2}^1}{1!}(x - x_1) + \frac{Q_{1,3}^2}{2!}(x - x_1)(x - x_2) + \frac{Q_{1,4}^3}{3!}(x - x_1)(x - x_2)(x - x_3), \tag{9}$$

obtained by a repeated application of equation (I).

In the general case the formation of a table of difference quotients involves a few laborious divisions; but these can be avoided by a proper choice of the spacing of the intervals of $x$. If, for instance, all of the intervals of $x$ are unity, then $\Delta_k x_i$, the average of any $k$ of them, will also be unity, and the rule for the formation of the difference quotients reduces to that of the formation of a table of differences. In that case the formulæ of Newton and Gauss are special cases of form (I).

These are by no means the only important special cases. Let us first consider the limiting case when one point, $(x_j, y_j)$, approaches the point, $(x_i, y_i)$. The limiting value of $Q_{i,j}^1$ would be $f'(x_i)$, if such a derivative existed, where $f(x) = y$ is the path along which the first point approached the second. The derivative may be regarded as a special case of the difference quotient, by extending the conception of the secant, as defined earlier, to take into account multiplicity. Whether regarded as a special case or as a limiting case, it may be used in formula (I) and on the right hand side of (II). Equation (II) does not hold in the special case when $Q_{1,n+2}^{n+1} = f^{(n+1)}(x_1)$, for if all the $(n+2)$ points are regarded as coincident, then the denominator is zero. In that case, however, $f^{(n+1)}(x_1)$ is part of the data and does not have to be computed.

Let us illustrate the case of two coincident points by a simple example. To form a polynomial which, when $x$ has the value 1, has the value 2 and its derivative the value 3; and when $x$ has the value 4, has the value 5. The table of difference quotients is below.

| $\Delta_2 x$ | $\Delta_1 x$ | $x$ | $y$ | $\Delta y$ | $Q^1$ | $\Delta Q^1$ | $Q^2$ |
|---|---|---|---|---|---|---|---|
| | | 1 | 2 | | | | |
| | 0 | | | 0 | 3 | | |
| $\frac{3}{2}$ | | 1 | 2 | | | $-2$ | $-\frac{4}{3}$ |
| | 3 | | | 3 | 1 | | |
| | | 4 | 5 | | | | |

Then

$$y = 2 + (x-1)3 + (x-1)^2\left(-\frac{4}{3}\right)\left(\frac{1}{2!}\right).$$

In the special case when all the points are coincident, the difference quotients are derivatives, and the polynomial one obtains from equation (I), is exactly the polynomial one would obtain using the same number of terms in the Taylor expansion.

Suppose we had an indefinite integral tabulated to a large number of decimal places, and wished to interpolate. For the sake of argument, suppose the table of differences would not converge rapidly enough. It often happens that we are able to compute the derivative of a function represented by an indefinite integral quite readily. It appears intuitively, that if we could in some way use to advantage the values of the derivative the process would converge more rapidly.

In the case under supposition, we know the values of a function and its derivative for equally spaced values of the argument. It is as if we knew the value of the function for equally spaced pairs of points, the two points of each pair being coincident. The difference quotient at any pair of coincident points is given to us as the derivative, and all the other difference quotients may be found by formula (II). There are so many simplifications in the process, that I will submit an example.

To find the value of $\log_{10} 31.029971$ to ten places. First we notice that it is easy to compute the derivative:

$$\frac{d}{dx}\log_{10} x = \frac{1}{x}\log_{10} e = \frac{0.4342944819}{x}.$$

We tabulate a few nearby logarithms, and the corresponding values of $\frac{0.4342944819}{x}$, as follows:

| $x$ | $\log x$ | derivative |
|---|---|---|
| 29 | 1. 46239 79979 | 0. 01497 56718 |
| 30 | 1. 47712 12547 | 0. 01447 64827 |
| 31 | 1. 49136 16938 | 0. 01400 94994 |
| 32 | 1. 50514 99783 | 0. 01357 17026 |
| 33 | 1. 51851 39399 | 0. 01316 04389. |

The table may be arranged as follows:

| $x$ | $y = \log x$ | $Q^1$ | $\frac{1}{2}Q^2$ | $\frac{1}{2}\Delta Q^2$ |
|---|---|---|---|---|
| 29. | 1. 46239 79979 | 0. 01497 56718 | $-0.$ 00025 24150 | 0. 00000 56409 |
| 29. | 1. 46239 79979 | 1472 32568 | 24 67741 | 1 07305 |
| 30. | 1. 47712 12547 | 1447 64827 | 23 60436 | 51039 |
| | | 1424 04391 | 23 09397 | 97248 |

| $x$ | $y = \log x$ | $Q^1$ | $\frac{1}{2}Q^2$ | $\frac{1}{2}\Delta Q^2$ |
|---|---|---|---|---|
| 31. | 1. 49136 16938 | 0. 01400 94994 | −0. 00022 12149 | 0. 00001 46330 |
| | | 1378 82845 | 21 65819 | 88409 |
| 32. | 1. 50514 99783 | 1357 17026 | 20 77410 | 42183 |
| | | 1336 39616 | 20 35227 | |
| 33. | 1. 51851 39399 | 1316 04389 | | |

| $\frac{1}{3}Q^3$ | $\frac{1}{4}Q^4$ | $\frac{1}{6}\Delta Q^4$ | $\frac{1}{8}Q^5$ |
|---|---|---|---|
| 0. 00001 12818 | −0. 00000 05513 | 0. 00000 00286 | 0. 00000 00858 |
| 1 07305 | 5227 | 396 | 792 |
| 1 02078 | 4830 | 243 | 729 |
| 97248 | 4588 | 336 | 672 |
| 92660 | 4251 | 207 | 621 |
| 88409 | 4043 | | |
| 84366 | | | |

The entries in the column headed $y$ are the data as copied from some table. Notice that the second entry is the same as the first, for they are the values of the function for two coincident values of $x$. This repetition is superfluous, and may be omitted. The column headed $Q^1$ contains the difference quotients at every two consecutive points. The first two values of $x$ are coincident, so the first difference quotient is copied from the table of derivatives. The second and third values of $x$ are a unit apart, so the difference quotient is the difference. And so on down the column, alternate entries are copied from the table of derivatives, and the others are obtained as differences of $y$.

In order to obtain $Q^2$, formula (II) tells us to divide the difference of two consecutive entries in the column of first order difference quotients by $\frac{1}{2}$. If you have a set of intervals, alternate ones zero and one, the average of any two consecutive ones of them will be $\frac{1}{2}$. Let us omit dividing by this factor, and thereby obtain $\frac{1}{2}Q^2$ instead of $Q^2$. To obtain the entries in the column headed $\frac{1}{2}\Delta Q^2$, we difference those of the column before. Formula (II) says that in order to obtain $Q^3$ we must divide the first $\Delta Q^2$ by 1/3, and the second by 2/3. The differences of $x$ are 0, 1, 0, 1, etc., and the average of three consecutive intervals will be 1/3 and 2/3 alternately. In order to avoid labor, we multiply every other one by 2, for they look smaller than the others, and thereby in each case obtain $\frac{1}{3}Q^3$ instead of $\frac{1}{2}Q^3$. Absorb these factors into the difference quotients, denoting the pseudo difference quotients by $Q_1^n$. The factorial divisors in the terms of the polynomial will be changed, as follows:

$$F(t) = y_1 + \frac{tQ_1}{(0!)^2(1!)} + \frac{t^2Q_1^2}{(0!)^2(1!)^2} + \frac{t^2(t-1)Q_1^3}{(1!)^2(2!)} + \frac{t^2(t-1)^2Q_1^4}{(1!)(2!)^2} + \cdots$$

$$+ \frac{t^2(t-1)^2(t-2)^2\cdots(t-n+1)^2(t-n)}{(n!)^2(n+1)!}\,Q_1^{2n+1} \tag{10}$$

$$+ \frac{t^2(t-1)^2(t-2)^2\cdots(t-n)^2}{(n!)((n+1)!)^2}\,Q_1^{2n+2}, \qquad (t = x - x_1).$$

We obtain in this particular example the polynomial:

$$y = 1.4913616938 + 0.0140094994t - 0.000221212149t^2 + \frac{t^2(t-1)}{2}0.0000092660 - \frac{t^2(t-1)^2}{4}0.0000004251 + \cdots.$$

To obtain log 31.029971, set $t = 0.029971$; adding the various terms of the polynomial, we obtain the result

$$\begin{array}{r} 1.\ 49136\ 16938 \\ 41\ 98787 \\ -1995 \\ -41 \\ -1 \\ \hline 1.\ 49178\ 13688 = \log 31.029971. \end{array}$$

Although the definition given above for the secant is geometrical, it could just as well be stated analytically. The work which I have carried out for reals could be extended into the domain of imaginaries.

When one has obtained the interpolatory polynomial, the secant, it may be asked, "How near will this come to giving the true value of the function throughout a certain interval?"

In equation (I), set $x = x_{n+2}$, and replace $P_{n+2}(x_{n+2};\ 1, n+2)$ by its equivalent value $f(x_{n+2})$. Considering $x_{n+2}$ as the independent variable, and calling it $x$ for simplicity, we get the equation:

$$\text{(III)} \quad f(x) \equiv P_n(x;\ 1, n+1) + \{Q^{n+1}_{1,\,n+1}(x)\}(x - x_1)(x - x_2)\cdots(x - x_{n+1})\frac{1}{(n+1)!}$$

where $Q^{n+1}_{1,\,n+1}(x)$ is the difference quotient taken for the $(n+1)$ fixed values $x_1, x_2, \cdots, x_{n+1}$, and the one variable value, $x$. It is a function of $x$.

A *remainder* is that which has to be added to an approximation in order to give the true result.

Let

$$f(x) = P_n(x;\ 1, n+1) + R(x).$$

Then

$$\text{(IV)} \qquad R(x) = \frac{1}{(n+1)!}\{Q^{n+1}_{1,\,n+1}(x)\}(x - x_1)(x - x_2)\cdots(x - x_{n+1}).$$

We have determined $R(x)$ in terms of $Q^{n+1}_{1,\,n+1}(x)$, and so let us study the latter. Let us examine its derivative. By the use of formula (II), we have

$$\left.\frac{dQ^{n+1}_{1,\,n+1}(x)}{dx}\right|_{x=x_0} = \operatorname*{Lim}_{x_{n+2}\to x_0}\frac{Q^{n+1}_{1,\,n+2} - Q^{n+1}_{0,\,n+1}}{x_{n+2} - x_0} = \operatorname*{Lim}_{x_{n+2}\to x_0}\frac{Q^{n+2}_{0,\,n+2}}{n+2}.$$

In general there is no limit, but let us suppose $f'(x)$ exists at $x = x_0$. Then the secant at the points $x_0, x_1, \cdots, x_{n+2}$ approaches a limiting secant as $x_{n+2}$ approaches $x_0$, namely the secant at the points $x_0, x_1, \cdots, x_{n+1}$, which has contact

at $x_0$. Its derivative of order $(n+2)$, the difference quotient, has a limiting value also, and is denoted thus: $Q^{n+2}_{0,0,n+2}$. Then we have

$$\text{(V)} \qquad \left.\frac{dQ^{n+1}_{1,n+1}(x)}{dx}\right|_{x=x_0} = \frac{Q^{n+2}_{0,0,n+2}}{n+2}.$$

Proceeding in this way one can determine the properties of $Q^{n+1}_{1,n+1}(x)$ from those of $f(x)$, and in that way can obtain the oscillation of the remainder for any interval. If the problem is to determine the derivative of $f(x)$, one differentiates in formula (III), and obtains the expression for the remainder in that case in terms of $Q^{n+1}_{1,n+1}(x)$ and its derivative, which are already known.

In this way, the difference quotient enables us to expand an arbitrary function, analytic or merely continuous, real or complex, at points to suit the peculiar needs of the problem. Moreover, in any special case, in which a simpler and less elastic method of expansion can be used, the expansion in terms of difference quotients reduces automatically to that method. The remainder can be expressed explicitly in terms of the difference quotient, not as in the case of a remainder in terms of a derivative at some unknown point, but taken at the points about which we are expanding and the one additional point $x$, the point at which we want the remainder.

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## A THEOREM IN THE GEOMETRY OF THE TRIANGLE.

By J. W. CLAWSON, Ursinus College.

It is the purpose of this paper to state a general theorem of which the well-known Wallace (or Simson[1]) line theorem is a very special case; to point out the result of a polar reciprocation; to mention some special cases of the theorem; and to give an application to the geometry of the quadrilateral.

1. *Theorem.*[2] $A_1A_2A_3$ is a triangle, $a_i$ the side opposite $A_i$, $(i = 1, 2, 3)$, $P$ a point on the circumference of the circumscribed circle[3] $C$, $Q$ any point in the plane. Let $A_iQ$ cut the circle[3] at $U_i$; $PU_i$ intersects $a_i$ at $L_i$. Then $L_1, L_2, L_3, Q$ are collinear on $l$. Let us call this line the "$P$-transversal of $Q$" with respect to the triangle $A_1A_2A_3$.

The theorem follows at once from Pascal's Theorem for, since $A_2A_3A_1U_1PU_2$, $A_1A_2A_3U_3PU_1$ are inscribed hexagons, $L_1, L_2, Q$ and $L_3, L_1, Q$ are collinear.

*Cor.* 1.—If a transversal cut the side $a_i$ of a triangle at $L_i$, and $P$ be any point on the circumcircle, and if $PL_i$ cut the circle at $U_i$, then $A_1U_1, A_2U_2, A_3U_3$ are concurrent at $Q$, a point on the transversal.

*Cor.* 2.—If a transversal cut the side $a_i$ of a triangle at $L_i$, if $Q$ is any point

[1] For references, see this MONTHLY, 1916, p. 61.

[2] It would be strange if this simple theorem were new; yet the writer has not found it recorded. A special case is given in Hatton's *Projective Geometry*, 1913, Ex. p. 164.

[3] Or conic.

on the transversal, and if $A_iQ$ cut the circumcircle at $U_i$, then $U_1L_1$, $U_2L_2$, $U_3L_3$ are concurrent at $P$, a point on the circumcircle.

2. Now let the polar reciprocal of the figure be found with respect to any circle having $P$ for center (Fig. 1). The sides of the triangle $A_1A_2A_3$ or $T$ reciprocate into the vertices of a triangle $A_1'A_2'A_3'$ or $T'$, which is inversely similar to $T$; and $A_1'$, $A_2'$, $A_3'$, $P$ lie on a circle $C'$. For[1]

$$\measuredangle A_1'PA_3' = \measuredangle A_3A_2A_1 = \measuredangle A_3PA_1 = \measuredangle A_1'A_2'A_3'.$$

If $A_1'U_1'$ is the reciprocal of $L_1$, $\measuredangle A_3'A_1'U_1' = \measuredangle A_2PL_1 = \measuredangle A_2A_1U_1$. Similarly, if $A_2'U_2'$ is the reciprocal of $L_2$, $\measuredangle A_1'A_2'U_2' = \measuredangle A_3PL_2 = \measuredangle A_3A_2U_2$. Hence $Q'$, the reciprocal of $l$, is the point in $T'$ isogonally conjugate to the point which corresponds to $Q$ in the inverse similarity of two triangles.

It is clear that $A_iU_i$ reciprocates into $L_i'$, the point where $PU_i'$ intersects $a_i'$; and that $Q$ reciprocates into $l'$, the $P$-transversal of $Q'$ with respect to $T'$.

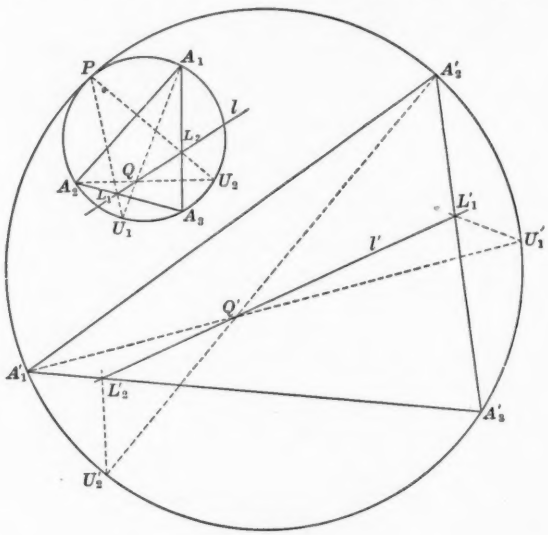


FIG. 1.

In particular, if $Q$ is at the in-center or at one of the ex-centers of $T$, its reciprocal is the $P$-transversal of the in-center or the corresponding ex-center of $T'$.

3. If the point $Q$ is at infinity,[2] it is easily shown by elementary geometry, remembering that $A_1U_1$, $A_2U_2$, $A_3U_3$ are parallel, that $\measuredangle PL_1A_2 = \measuredangle PL_2A_3 = \measuredangle PL_3A_1$. The line is in this case one of the family of lines described by

[1] The notation is that of R. A. Johnson for "directed angles" (this MONTHLY, 1917, p. 101), $\measuredangle BAC$ meaning the positive angle through which the line $AB$, taken as a whole, must be rotated, to coincide with the line $AC$, taken as a whole.

[2] This is the case given in Hatton, *loc. cit.*

Poncelet in his *Propriétés Projectives*, 1822, § 468, as generalizations of the Wallace line.[1]

*Cor.* If $l$ is a transversal cutting the side $a_i$ of a triangle in $L_i$ and if $A_iU_i$ is drawn parallel to it cutting the circumcircle at $U_i$, then $L_1U_1$, $L_2U_2$, $L_3U_3$ concur at a point $P$ on the circumcircle. Moreover $\measuredangle PL_1A_2 = \measuredangle PL_2A_3 = \measuredangle PL_3A_1$.[2]

Now in the general case, $\measuredangle PL_1A_2 = \measuredangle Q'A_1'P$ and $\measuredangle PL_2A_3 = \measuredangle Q'A_2'P$. Hence, in the special case under consideration, $\measuredangle Q'A_1'P = \measuredangle Q'A_2'P$. Hence $Q'$ is on the circle $C'$, $l'$ being in fact the chord $PQ'$.

Since the reciprocals of $L_1$, $L_2$, $L_3$ concur at a point on the circle $C'$, the line $l$ touches a parabola, focus $P$, which touches the sides of the triangle $T$. Hence Poncelet's lines envelope this parabola.[3]

In particular, if $Q'$ is the point diametrically opposite to $P$ in $C'$, $l$ will evidently be the pedal (or Wallace) line of the point $P$ with respect to the triangle $T$. That is, if $Q$ is at infinity in the direction of the directrix of the parabola of Steiner, $l$ is the pedal line.

Since, in this last case, $Q'$ and the circumcenter of $T'$ are collinear with $P$, their reciprocals, the pedal line of $P$ and the $P$-transversal of $H$, where $H$ is the orthocenter of $T$, are parallel, as may easily be proved directly. This $P$-transversal of $H$ is then the directrix of Steiner's parabola.

Again, if $Q$ is the point isogonally conjugate to $P$ with respect to $T$, since

$$\measuredangle A_2A_3A_1 = \measuredangle A_2PA_1 = \pi - \measuredangle A_1A_2P - \measuredangle PA_1A_2 = \pi - \measuredangle QA_2A_3 - \measuredangle A_3A_1Q,$$

$Q$ is at infinity.[4]

Now $\measuredangle A_3A_1Q = \measuredangle PA_1A_2 = \measuredangle PU_1A_2$. But $\measuredangle U_1A_1A_3 = \measuredangle U_1A_2A_3$. Moreover, if $l$ is the pedal line, $\measuredangle PU_1A_2$ and $\measuredangle U_1A_2A_3$ are complementary. Hence $A_1Q$ is perpendicular to $A_1U_1$. But in this case (in fact for any of Poncelet's lines) $A_1U_1$ is parallel to $l$. Hence $A_1Q$ is perpendicular to the pedal line of $P$.[5] Then $PQ$ is the axis of the Steiner parabola when $Q$ is the point isogonally conjugate to $P$. It follows that the $P$-transversal of $Q$ in this case, a tangent to the Steiner parabola at infinity, must be the line at infinity. This can be proved independently by showing that in this case $PU_i$ is parallel to $a_i$.

4. Let $A_{12}$, $A_{34}$; $A_{13}$, $A_{24}$; $A_{14}$, $A_{23}$ (Fig. 2), be opposite vertices of a complete quadrilateral. Let $P$ be its Wallace point,[6] common to the circumcircles of the four triangles $T_1$, $T_2$, $T_3$, $T_4$, where $T_1$ is $A_{23}A_{34}A_{24}$. It is evident that the polar reciprocal of the quadrilateral with respect to any circle, center $P$, is a cyclic quadrangle whose vertices $L_1'$, $L_2'$, $L_3'$, $L_4'$ are the reciprocals of the four sides of the quadrilateral. The circle $C'$, circumscribing the quadrangle, passes through $P$.

[1] Mackay, *Proc. Edin. Math. Soc.*, 1890–1891.

[2] Boyman (*Archiv der Math. u. Phys.*, 1849, 13, p. 364) gives a theorem equivalent to this corollary.

[3] Steiner, Gergonne's *Annales*, XIX, 1828.

[4] Casey, *Sequel to Euclid*, 1886, Ex. 4, p. 167.

[5] Lachlan, *Modern Pure Geometry*, 1893, Ex. 4, p. 68.

[6] Mackay, *loc. cit.*

Now it is easily proved[1] that the 16 in- and ex-centers of the four triangles formed by taking the vertices of the quadrangle in threes lie four by four on four lines parallel to a bisector of the angle between $L_1'L_3'$ and $L_2'L_4'$ and also four by four on four lines perpendicular to these. In the figure, $J_1'$, $J_{21}'$, $J_{34}'$, $J_4'$ are shown lying on one of these lines.

By reciprocation, remembering the fact asserted in the last paragraph of 2, the 16 $P$-transversals of the in- and ex-centers of the four triangles $T_i$ ($i = 1, 2, 3, 4$), are concurrent in fours in four points which all lie on a line through $P$

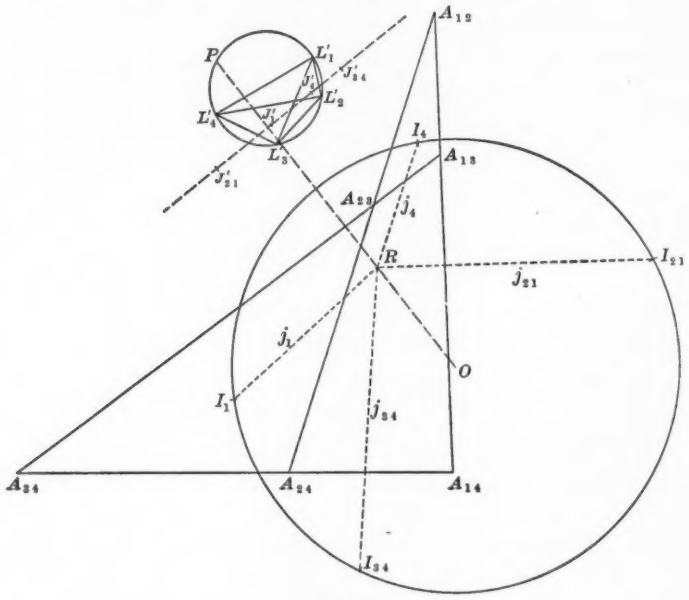


FIG. 2.

bisecting the angle $A_{12}PA_{34}$. One of these points, at which meet the $P$-transversals of $I_1$, $I_{21}$, $I_{34}$, $I_4$ is shown in the figure at $R$.

It may be added that these two lines ($PR$ and the line perpendicular to it at $P$) are the lines otherwise defined by Steiner,[2] as determined each by the centers of four circles on each of which lie four of the 16 in- and ex-centers of the triangles $T$. One of these circles, determined by $I_1$, $I_{21}$, $I_{34}$, $I_4$, Center $O$, is shown in the figure. The writer has been able to prove this fact only by making use of a long, circuitous proof based on that given by Mention in *Nouv. Ann. de Math.*, 1862, p. 76, proving that the line $OP$ is also a bisector of the angle $A_{12}PA_{34}$. It seems as though a simpler proof might be found.

[1] Neuberg, *Mathesis*, 6, 1906, p. 14. This MONTHLY, 1917, p. 430.
[2] Gergonne's *Annales*, 1828, XVIII, p. 302.

# ON AN ELEMENTARY PROBLEM OF CLOSURE ON AN EQUILATERAL HYPERBOLA.

By ARNOLD EMCH, University of Illinois.

In a recent number of the *Nouvelles Annales de Mathématiques*[1] Professor P. Appell proposes the following problem concerning a system of quadruples of points on an equilateral hyperbola:

Let $A_1$, $B_1$, $C_1$, $D_1$ be four points on an equilateral hyperbola; the altitudes of the triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$ intersect respectively in their ortho-centers $A_2$, $B_2$, $C_2$, $D_2$, which also lie on the curve. Thus, from the four points $A_1$, $B_1$, $C_1$, $D_1$, four new points $A_2$, $B_2$, $C_2$, $D_2$, are derived; in the same manner from these four other points $A_3$, $B_3$, $C_3$, $D_3$ are obtained, . . . and so forth. Under what conditions does the quadruple $A_nB_nC_nD_n$ coincide entirely, or in part, with $A_1B_1C_1D_1$?

According to Appell's suggestion one may choose $xy = 1$ as the equation of the equilateral hyperbola and represent it parametrically in the form

$$x = e^t, \qquad y = e^{-t}, \tag{1}$$

with $t$ as the parameter. If $b_1$, $c_1$, $d_1$, $a_2$ are the abscissas of $A_1$, $C_1$, $D_1$ and the orthocenter of $A_1C_1D_1$, respectively, then, as is well known, $a_2b_1c_1d_1 = -1$. Hence, when $\alpha_1$, $\beta_1$, $\gamma_1$, $\delta_1$; $\alpha_2$, $\beta_2$, $\cdots$ are the parameters of $A_1$, $B_1$, $C_1$, $D_1$; $A_2$, $B_2$, $\cdots$, we have, when the congruence sign $\equiv$ refers throughout to modulo $2i\pi$ (where $i = \sqrt{-1}$),

$$\begin{aligned}
\alpha_{p+1} + \beta_p + \gamma_p + \delta_p &\equiv i\pi,\\
\beta_{p+1} + \gamma_p + \delta_p + \alpha_p &\equiv i\pi,\\
\gamma_{p+1} + \delta_p + \alpha_p + \beta_p &\equiv i\pi,\\
\delta_{p+1} + \alpha_p + \beta_p + \gamma_p &\equiv i\pi.
\end{aligned} \tag{2}$$

Putting $S_n = \alpha_n + \beta_n + \gamma_n + \delta_n$, from (2) is found

$$S_{p+1} + 3S_p \equiv 0, \tag{3}$$

and

$$\begin{aligned}
\alpha_{p+1} - \alpha_p + S_p &\equiv i\pi,\\
\beta_{p+1} - \beta_p + S_p &\equiv i\pi,\\
\gamma_{p+1} - \gamma_p + S_p &\equiv i\pi,\\
\delta_{p+1} - \delta_p + S_p &\equiv i\pi.
\end{aligned} \tag{4}$$

This is as far as Appell's suggestion goes. From (3) and (4) by recurrence

[1] Vol. XVIII, pp. 41–42 (February, 1918).

$\alpha_{n+1}$ may be expressed in terms of $\alpha_1$ and $S_1$ as

$$\alpha_{n+1} = \alpha_1 - \frac{1-(-3)^n}{4} S_1 + n'i\pi, \tag{5}$$

where $n'$ is even or odd with $n$.

Similar formulas are obtained for the parameters $\beta_{n+1}$, $\gamma_{n+1}$, $\delta_{n+1}$ of the points $B_{n+1}$, $C_{n+1}$, $D_{n+1}$. The points $A_{n+1}$, $B_{n+1}$, $C_{n+1}$, $D_{n+1}$ coincide in the same order with $A_1$, $B_1$, $C_1$, $D_1$, when

$$-\frac{1-(-3)^n}{4} \cdot S_1 + n'i\pi \equiv 0, \tag{6}$$

which, in case of real points, is possible only when either

I $$S_1 = \alpha_1 + \beta_1 + \gamma_1 + \delta_1 \equiv i\pi,$$

or

II $$S_1 = \alpha_1 + \beta_1 + \gamma_1 + \delta_1 \equiv 0.$$

Clearly, in the first case, the quadruple $A_2B_2C_2D_2$ coincides in the same order with $A_1B_1C_1D_1$, and is therefore closed in itself. For case II, from (4) we find $\alpha_2 = \alpha_1 - S_1 + (2\lambda+1)i\pi$, or $\alpha_2 = \alpha_1 + (2\lambda+1)i\pi$; similarly

$$\beta_2 = \beta_1 + (2\lambda+1)i\pi, \qquad \gamma_2 = \gamma_1 + (2\lambda+1)i\pi, \qquad \delta_2 = \delta_1 + (2\lambda+1)i\pi.$$

From this follows that the abscissas of $A_2$, $B_2$, $C_2$, $D_2$ are in the same order opposite in sign to those of $A_1$, $B_1$, $C_1$, $D_1$, *i. e.*, that the quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are congruent and in central symmetry with respect to the center of the hyperbola.

Since the condition for four concyclic points $A_1$, $B_1$, $C_1$, $D_1$,

$$\begin{vmatrix} e^{2\alpha_1}+e^{-2\alpha_1} & e^{\alpha_1} & e^{-\alpha_1} & 1 \\ e^{2\beta_1}+e^{-2\beta_1} & e^{\beta_1} & e^{-\beta_1} & 1 \\ e^{2\gamma_1}+e^{-2\gamma_1} & e^{\gamma_1} & e^{-\gamma_1} & 1 \\ e^{2\delta_1}+e^{-2\delta_1} & e^{\delta_1} & e^{-\delta_1} & 1 \end{vmatrix}$$

$$= \{e^{-2(\alpha_1+\beta_1+\gamma_1+\delta_1)} - e^{-(\alpha_1+\beta_1+\gamma_1+\delta_1)}\} \cdot \begin{vmatrix} e^{3\alpha_1} & e^{2\alpha_1} & e^{\alpha_1} & 1 \\ e^{3\beta_1} & e^{2\beta_1} & e^{\beta_1} & 1 \\ e^{3\gamma_1} & e^{2\gamma_1} & e^{\gamma_1} & 1 \\ e^{3\delta_1} & e^{2\delta_1} & e^{\delta_1} & 1 \end{vmatrix} = 0$$

is evidently satisfied in case II, the original quadrangle $A_1B_1C_1D_1$, and consequently also the quadrangle $A_2B_2C_2D_2$ is concyclic. This follows also from the fact that the product of the abscissas of four concyclic points on the hyperbola (1) is $+1$. From (4) follows further that

$$\alpha_3 = \alpha_1 + 2k\pi, \qquad \beta_3 = \beta_1 + 2k\pi, \qquad \gamma_3 = \gamma_1 + 2k\pi, \qquad \delta_3 = \delta_1 + 2k\pi,$$

so that $A_3B_3C_3D_3$ coincides in the same order with $A_1B_1C_1D_1$. Hence in case II the closed series contains two quadruples. Outside of I and II there are no other cases of closed series of *real* quadruples.

The result of case II may be stated in the

THEOREM: *Four concyclic points $A_1$, $B_1$, $C_1$, $D_1$ on an equilateral hyperbola form four triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$, whose orthocenters $A_2$, $B_2$, $C_2$, $D_2$ are also concyclic, and in central symmetry with $A_1B_1C_1D_1$, with respect to the center of the hyperbola. The orthocenters of $B_2C_2D_2$, $C_2D_2A_2$, $D_2A_2B_2$, $A_2B_2C_2$ coincide in the same order with $A_1$, $B_1$, $C_1$, $D_1$.*

The fact that $A_2$, $B_2$, $C_2$, $D_2$ are concyclic is independent of the fact that the points $A_1B_1C_1D_1$ are on an equilateral hyperbola. Necessary and sufficient condition is that $A_1$, $B_1$, $C_1$, $D_1$ are concyclic. It is always possible to pass an equilateral hyperbola through the points of a proper quadrangle. In fact, the pencil of conics through such a quadrangle intersects the line at infinity in an involution. Any finite point joined to this involution determines an involutory pencil, which contains at least one rectangular pair whose directions are parallel to the asymptotes of the corresponding hyperbola of the pencil. This hyperbola is therefore equilateral.

The foregoing theorem concerning a concyclic quadrangle is not new. It was first stated without proof by Steiner in a foot-note.[1] A purely geometrical proof was first given by Heinen.[2] Later on an analytic proof was published by Greiner.[3]

The question whether $A_{n+1}$, $B_{n+1}$, $C_{n+1}$, $D_{n+1}$ may coincide in the same order, say, with $B_1D_1A_1C_1$ must be answered in the negative, if only real solutions are considered. According to (5) and the three similar equations such a condition would give for $\alpha_1$, $\beta_1$, $\gamma_1$, $\delta_1$ fractional values of $i\pi$, and, therefore no proper quadrangle on the hyperbola. Coincidences of less than four points may occur, but as they seem not of sufficient geometric interest, they will not be considered in this note.

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## REMARKS ON A PREVIOUS ARTICLE.

By NATHAN ALTSHILLER, University of Oklahoma.

In connection with my article "On the I-centers of a Triangle"[1] Prof. J. W. Clawson kindly calls my attention to the fact that the propositions (11) and (13) are known. They were proved in 1906 by the well-known Belgian mathematician, Prof. J. Neuberg, of the University of Liege.[5] Prof. Neuberg states in his article that these propositions were published before without proof.[5]

I arrived at these results in the early summer of 1917 while giving a course in "College Geometry" at the University of Oklahoma, Summer Session. The library facilities at my command were inadequate for a satisfactory biblio-

[1] *Annales de Mathématiques Pures Appliquées*, Vol. 19, p. 43 (1828).
[2] *Journal für die reine und angewandte Mathematik*, Vol. 3, p. 291 (theorem 9) (1828).
[3] *Archiv der Mathematik und Physik*, Vol. 60, p. 184 (1877).
[4] This MONTHLY, June, 1918, pp. 241–246.
[5] *Mathesis*, 1906, pp. 14–17.
[6] These theorems were also discovered independently by J. V. Morley and published as problems in Volume 24 of this MONTHLY, pages 124 and 430.

graphical investigation. It would seem however that these properties are not well known, since just recently the theorem (11) has been proposed for proof in as serious a journal as the *Nouvelles Annales de Mathématiques*.[1]

It is noteworthy that the sides of the rectangle (11) are proved by Prof. Neuberg to be parallel to the bisectors of the angles formed by the diagonals of the quadrilateral; V. Thébault states that these sides are equally inclined on the sides of the given quadrilateral, while in my paper they are shown to be parallel to the lines joining the mid-points of the arcs subtended by the opposite sides of the given quadrilateral on its circumcircle (compare sections 10, 11). Prof. Clawson adds that they are parallel to the bisectors of the angles formed by any pair of opposite sides of the given quadrilateral.

Thus comes to light an interesting and rather involved property of the inscriptible quadrilateral, worthy perhaps of a direct proof.

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## BOOK NOTICES.

Edited by W. H. BUSSEY, University of Minnesota.

A circular advertising the new book on *Unified Mathematics* by KARPINSKI, BENEDICT and CALHOUN says that "particular attention is paid to problems dealing with projectiles, and the 'mil,' the artillery unit of angular measurement, is carefully explained." But the reader will look in vain for the word "mil" in the index of the book; and he will look in vain in the paragraphs on angles and angular measurement where the degree and radian are defined. However if he is bound to know what a "mil" is and searches further he will be rewarded when he finds Ex. 11 on page 114 which reads: "In the artillery service angles are measured in 'mils'; a 'mil' is defined as 1/6,400 of a complete revolution. Compute the value in radians of one mil." Of course all this is in no sense a real criticism of the book, which was not written primarily for men of the S. A. T. C. The book contains 522 pages. It is supposed to be a course in elementary mathematics adapted to the needs of the freshmen students in the ordinary college or technical school course. According to the preface the material includes the work commonly covered in the past in separate courses in college algebra, trigonometry and analytical geometry. But there is no chapter on Permutations, Combinations and the more simple elements of Probability, and there is no mention of these topics in the index. As one might expect from the fact that Prof. Karpinski is known to be interested in the history of mathematics, the book abounds in historical notes and references. The book is published by D. C. Heath and Co.

Unified mathematics seems to be making its way. In addition to the book just mentioned, several other books on correlated mathematics have recently been published or are about to be published.

[1] V. Thébault, *N. A. M.*, August, 1917, p. 319.

The McGraw-Hill Book Co. has published a new and revised edition of the well known *Elementary Mathematical Analysis* written by Professor CHARLES S. SLICHTER of the University of Wisconsin.

Another unified mathematics book is *Freshmen Mathematics* by Professor WILLIAM R. RANSOM, of Tufts College, published by Longmans, Green and Co. It is a book of 285 pages, much less extensive than the two books just mentioned. "The chief features of the book are its brevity, the breadth and simplicity of its methods, its selection of subject matter for utility and interest rather than for mathematical completeness, and the careful preparation of problem material." Among other novel features it contains a short chapter entitled "Trigonometry in three dimensions" which begins with an explanation of the theory of the sun-dial and ends with longitude problems.

Ginn and Company have just published an *Introduction to the Elementary Functions* by R. B. MCCLENON and WILLIAM J. RUSK, of Grinnell College. It is an attempt to solve the problem of the first year collegiate course in mathematics. The idea of functionality is the unifying principle. The last chapter is a 28 page introduction to the differential calculus. There is no work on integral calculus, although the authors "firmly believe that this topic should eventually be included in the first year course."

Ginn and Co. are about to publish a book called *General Mathematics*, by Mr. RALEIGH SCHORLING, of the Lincoln School, Teachers' College, Columbia University, and Mr. W. D. REEVE, of the University High School, University of Minnesota. It is the first of a series of four volumes on mathematics for high school students. The first volume is for first year high school students.

At the annual meeting of the Association in December, 1917, many mathematicians who had never studied descriptive geometry were awakened to an interest in the subject by Professor Roever's paper on descriptive geometry. Those who have not yet followed that inspiration to study the subject may be interested in the fifth edition of *Descriptive Geometry* by W. L. AMES and CARL WISCHMEYER, of Rose Polytechnic Institute, recently published by the McGraw-Hill Book Company. It is a small book of 112 pages.

There has recently been issued by the Bureau of Education at Washington a Bulletin on *The Training of Teachers of Mathematics for the Secondary Schools of the Countries Represented in the International Commission on the Teaching of Mathematics*. This Bulletin has been prepared by Professor R. C. ARCHIBALD, of Brown University. It is a work of nearly three hundred pages, giving in great detail the requirements set by the various governments for a teacher of secondary mathematics. The Bureau of Education has a limited number of copies of this Bulletin which it can send to those who are particularly interested in the work. After this limited number has been exhausted, copies can be obtained from the Superintendent of Documents, Government Printing Office at Washington, D. C., at 30 cents per copy.

Contents—Introduction, pp. 3–4; chapter I: Australia, 5–14; II: Austria, 15–27; III: Belgium, 28–38: IV: Denmark, 39–44; V: England, 45–60, VI: Finland, 61–65, VII: France, 66–

76; VIII: Germany, 71–129; IX: Hungary, 130–137; X: Italy, 138–143; XI: Japan, 144–152; XII: The Netherlands, 153–157: XIII: Romania, 158–159; XIV: Russia, 160–167; XV: Spain, 168–170; XVI: Sweden, 171–190; XVII: Switzerland, 191–199; XVIII: The United States, 200–211; XIX: Summary and comparative remarks, 212–230. Appendices, *A*, *B*, England: Cambridge local examinations senior students; Oxford and Cambridge schools; examination board; University of London matriculation examinations; Entrance scholarship examination papers, Cambridge, 231–251; *C*, *D*, France: Concours for admission to the Ecole Normale Supérieure, and for the bourses de licence in 1913, Agrégation des sciences mathématiques, 252–266; *E*, Germany: Reifeprüfungen, Lehramtsprüfungen, 267–274; *F*, Japan: Examination questions, 275–278. Index, 279–289.

## QUESTIONS AND DISCUSSIONS.

Edited by U. G. MITCHELL, University of Kansas, Lawrence.

Three questions which, among others, have been standing for some time without having been answered are republished below in the hope that some of our readers may thereby be stimulated to send in suitable replies.

**15.** In the *Proceedings of the Royal Society of Edinburgh*, Vol. VII, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

(A partial reply to the above has been received showing that if $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$. Can some one show how the "easy proof" then follows?)

**21.** For the diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$$x = \quad 3, \quad 4, \quad 5, \quad 9, \quad 23, \quad 282, \quad 375, \quad 378661,$$
$$y = -2, \quad -1, \quad 2, \quad 4, \quad 8, \quad 43, \quad 52, \quad 5234.$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

**32.** In a discussion of the Peaucellier[1] Cell by analytic methods the following equations are obtained:

$$(1)\quad (x_2 - x_1)^2 + (y_2 - y_1)^2 - b^2 = 0; \qquad (2)\quad (x_3 - x_1)^2 + (y_3 - y_1)^2 - b^2 = 0;$$
$$(3)\quad (x_2 - X)^2 + (y_2 - Y)^2 - b^2 = 0; \qquad (4)\quad (x_3 - X)^2 + (y_3 - Y)^2 - b^2 = 0;$$
$$(5)\quad x_2^2 + y_2^2 - K^2 = 0; \qquad (6)\quad x_3^2 + y_3^2 - K^2 = 0;$$
$$(7)\quad x_1^2 + y_1^2 - 2cx_1 = 0.$$

The result of eliminating $x_1, y_1, x_2, y_2, x_3, y_3$ gives an equation of the first degree, which establishes that the linkage will trace a straight line. There are various ways of effecting this elimination.

1. What element of the situation is left unused by the following procedure in the elimination?
   (*a*) From equations (1), (3), (5) eliminate $x_2$ and $y_2$ and obtain an equation
   $$(8)\quad f_1(x_1, y_1) = 0.$$
   (*b*) From equations (2), (4), (6) eliminate $x_3$ and $y_3$ and obtain an equation
   $$(9)\quad f_2(x_1, y_1) = 0.$$
   (*c*) From equations (7), (8), (9) eliminate $x_1$ and $y_1$ and obtain the desired equation.
2. How should this procedure be supplemented to secure the result?

[1] If reference is made to the article on "Linkages" in the December, 1915, MONTHLY, by Mr. Leavens, the following coördinates may be applied to his figure: $O(0, 0)$; $C(c, 0)$; $P_1(x_1, y_1)$; $M(x_2, y_2)$; $M_1(x_3, y_3)$; $P_2(X, Y)$.

## NEW QUESTION.

**36.** A number of Discussions have been published in this department relating to cubic and biquadratic equations (cf. Vol. XXXV, p. 29, pp. 268–269 and 343–347; Vol. XXIV, pp. 136–137 and 436–439; Vol. XXIII, pp. 314–315). Below are given a number of questions sent in by Professor Harris Hancock of the University of Cincinnati which relate to the cubic and biquadratic and might, perhaps, more properly be proposed as problems were it not for the advantage to be gained, if possible, by treating them all, or at least several of them, in one discussion.

1. For what values of $n$ can $\cos 2\pi/n$ be expressed in the form $(a + \sqrt{b})/c$ where $a$, $b$ and $c$ are integers?

2. Write the biquadratic in the form $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$. Show that its reducing cubic may be expressed by means of a determinant of the third order which when expanded is

$$4y^3 - g_2 y - g_3 = 0,$$

where $g_2 = ae - 4bd + 3c^2$, and

$$g_3 = \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = ace - ad^2 - eb^2 - c^3 + 2bcd.$$

3. For the same biquadratic show that

$$a^3(x_0 + x_1 - x_2 - x_3)(x_0 + x_2 - x_1 - x_3)(x_0 + x_3 - x_1 - x_2) = 32(3abc - a^2d - 2b^3)$$

without making any use of symmetric functions.

4. If $x_0$, $x_1$, $x_2$ are the roots of a cubic and $D$ its discriminant, show that $x_1$ is a rational function of $x_0$ and $\sqrt{D}$. Derive a much simpler relation than that given in Serret, *Cours d'Algèbre Supérieure*, 5th ed., No. 511.

5. If $x_0$, $x_1$, $x_2$, $x_3$ are the roots of a biquadratic, $D$ its discriminant, $e_1$, $e_2$, $e_3$ the roots of the reducing cubic, show that $x_1$ is a rational function of $x_0$, $e_1$, $e_2$, $e_3$ and consequently also of $x_0$, $e_1$ and $\sqrt{D}$.

6. If the biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0 \tag{1}$$

has a double root, show that the reducing cubic (cf., for example, Burnside and Panton's *Theory of Equations*)

$$t^3 + 3Ht^2 + \left(3H - \frac{a^2I}{4}\right)t - G^2/4 = 0 \tag{2}$$

has a root in common with the cubic

$$8t^3 + 12Ht^2 + G^2 = 0; \tag{3}$$

and conversely, if (2) and (3) have a common root, then (1) has a double root.

## DISCUSSIONS.

### Concerning Haversines in Plane Trigonometry.

By G. W. Evans, Charlestown High School, Boston, Mass.

The work for the S. A. T. C. in American colleges has called attention to the use of haversines. It appears not to be generally known that plane trigonometry can be simplified, much as to teaching and somewhat as to computation, by the use of these ratios.

The haversine is defined as half the versed sine; that is,

$$\text{hav } A = \frac{1 - \cos A}{2} = \sin^2 \frac{A}{2}.$$

For the case where a right triangle is given by the hypotenuse $c$ and one side, $b$, nearly equal to $c$, there formerly appeared in elementary text-books the formula

$$\sin \tfrac{1}{2}A = \sqrt{\frac{c - b}{2c}}$$

but its application seemed to be not much insisted upon. In the form

$$\text{hav } A = \frac{c - b}{2c}$$

it is an obvious deduction from the definition of cos $A$ and there is no square root to be found.

In oblique triangles the case where two sides and the included angle are given and the case where the three sides are given have formulæ that require a considerable amount of detail in the way of proof. With haversines the proofs would be direct and simple.

From the fundamental formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

we obtain

$$\frac{1 - \cos A}{2} = \frac{a^2 - (b - c)^2}{4bc}.$$

From this equation, for the case where two sides and the included angle are given, we have the formula

I. $$a^2 - (b - c)^2 = 4bc \text{ hav } A.$$

For the case where the three sides are given, we have, from the same equation, the formula

II. $$\text{hav } A = \frac{(s - b)(s - c)}{bc}.$$

The following arrangement of work is suggested:

Example 1. Given $A = 94.39°$, $b = 55.12$, $c = 39.90$.

Formulæ: $a^2 - (b - c)^2 = 4bc \text{ hav } A$; $\sin B = \dfrac{b \sin A}{a}$.

| $A = 94° 23.4'$. | | Logarithms. | | | |
|---|---|---|---|---|---|
| $b =$ | 55.12 | | | $b$ | 1.7414 |
| $c =$ | 39.90 | | | $c$ | 1.6010 |
| $b - c =$ | 15.22 | | 1.1824 | hav $A$ | 9.7310 |
| $(b - c)^2 =$ | 231.6 | | 2.3648 | 4 | 0.6021 |
| $a^2 - (b - c)^2 =$ | 4737. | | | | 3.6755 |
| $a^2 =$ | 4969. | | 3.6963 | | |
| $a =$ | 70.50 | | 1.8482 | | |
| $A =$ | 94.39° | $\sin A$ | 9.9987 | $\sin A$ | 9.9987 |
| $B =$ | 51.24° | $\frac{1}{a}$ | 8.1518 | $\frac{1}{a}$ | 8.1518 |
| $C =$ | 34.36° | $b$ | 1.7414 | $c$ | 1.6010 |
| Check: | 179.99° | $\sin B$ | 9.8919 | $\sin C$ | 9.7515 |

Example 2. Given $a = 70.50$, $b = 55.12$, $c = 39.90$.

$$\text{Formula: hav } A = \frac{(s-b)(s-c)}{bc}.$$

Logarithms.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $a = 70.50$ | 1.8482 | | | $\frac{1}{a}$ | 8.1518 | $\frac{1}{a}$ | 8.1518 |
| $b = 55.12$ | 1.7414 | $\frac{1}{b}$ | 8.2586 | | | $\frac{1}{b}$ | 8.2586 |
| $c = 39.90$ | 1.6010 | $\frac{1}{c}$ | 8.3990 | $\frac{1}{c}$ | 8.3990 | | |
| $2s = 165.52$ | | | | | | | |
| $s = 82.76$ | | | | | | | |
| $s - a = 12.26$ | 1.0885 | | | | 1.0885 | | 1.0885 |
| $s - b = 27.64$ | 1.4415 | | 1.4415 | | | | 1.4415 |
| $s - c = 42.86$ | 1.6321 | | 1.6321 | | 1.6321 | | |
| | log hav | $A$ | 9.7312 | $B$ | 9.2714 | $C$ | 8.9404 |

Check: $94° 25' + 51° 13' + 34° 21' = 179° 59'$.

The work in example 2 very closely resembles the work in spherical trigonometry for the solution of the triangle in which three sides are given. The work in example 1 is of the same character as for the corresponding problem in spherical trigonometry; for example, where the latitude, longitude, and declination are used to compute the altitude for locating a Sumner line. Aside, however, from the advantage that this method of treatment may have in preparing for the work in spherical trigonometry, it has immediate advantage here in the fact that it bases the work in plane triangles on the sine formula for the first two cases, and on the cosine formula for the last two. The use of one or the other of the two haversine formulæ for checking the work in the first two cases makes the check somewhat more laborious than the original computation, but that is no reason why the check should not be insisted upon. On the other hand, the check in the last two cases is simple.

Five place tables of natural haversines and of logarithmic haversines, values for every 15″, are published in Bowditch's *American Practical Navigator* and in publication No. 200 of the U. S. Hydrographic Office; five-place tables for every minute in H. Jacoby's *Navigation* (Macmillan, 1918). The Harvard University Press printed, in the fall of 1918, a four-place table of logarithms of haversines,[1]

[1] Prepared by Mr. Evans, and occupying two of the four pages of a leaflet.

The term haversine was introduced by James Inman (1776–1859) in the third edition (1835) of his *Navigation and Nautical Astronomy for the Use of British Seamen*. A table of logarithms of haversines is there given. Inman was professor of navigation and nautical science in the Royal Naval College, Portsmouth. See *Dictionary of National Biography*. This note constitutes the reply to Question 2311 (by C. Wargny of Valparaiso) in *L'Intermédiaire des Mathématiciens*, April, 1902.—Editor.

from which the entries in the above examples are taken. It is to be hoped that such tables will be included in text-books, and made available at the examinations of the College Entrance Board.

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## PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS AND SOLUTIONS TO B. F. FINKEL, Springfield, Mo.

**2742. Proposed by S. A. COREY, Des Moines, Iowa.**

Establish the following algebraic identity without actually performing the indicated operations:

$$\begin{aligned}
&2(l_1l_2 + c_1l_3l_4 + c_2l_5l_6 + c_1c_2l_7l_8)(r_1r_2 + c_1r_3r_4 + c_2r_5r_6 + c_1c_2r_7r_8) \\
&\quad = (r_1l_1 - c_1r_3l_3 - c_2r_5l_5 + c_1c_2r_7l_7)(r_2l_2 - c_1r_4l_4 - c_2r_6l_6 + c_1c_2r_8l_8) \\
&\quad + (r_1l_2 - c_1r_3l_4 - c_2r_5l_6 + c_1c_2r_7l_8)(r_2l_1 - c_1r_4l_3 - c_2r_6l_5 + c_1c_2r_8l_7) \\
&\quad + c_1(r_1l_3 + r_3l_1 - c_2r_5l_7 - c_2r_7l_5)(r_2l_4 + r_4l_2 - c_2r_6l_8 - c_2r_8l_6) \\
&\quad + c_1(r_1l_4 + r_3l_2 - c_2r_5l_8 - c_2r_7l_6)(r_2l_3 + r_4l_1 - c_2r_6l_7 - c_2r_8l_5) \\
&\quad + c_2(r_1l_5 + c_1r_3l_7 + r_5l_1 + c_1r_7l_3)(r_2l_6 + c_1r_4l_8 + r_6l_2 + c_1r_8l_4) \\
&\quad + c_2(r_1l_6 + c_1r_3l_8 + r_5l_2 + c_1r_7l_4)(r_2l_5 + c_1r_4l_7 + r_6l_1 + c_1r_8l_3) \\
&\quad + c_1c_2(r_1l_7 - r_3l_5 + r_5l_3 - r_7l_1)(r_2l_8 - r_4l_6 + r_6l_4 - r_8l_2) \\
&\quad + c_1c_2(r_1l_8 - r_3l_6 + r_5l_4 - r_7l_2)(r_2l_7 - r_4l_5 + r_6l_3 - r_8l_1).
\end{aligned}$$

By assuming special relations between the constants involved show that the product of the sum of four squares by the sum of four squares equals the sum of four squares.

**2743. Proposed by DANIEL KRETH, Wellman, Iowa.**

In the right angle triangle $ABC$, right angle $C$, we have given on the hypotenuse the segments $AD = 15$, $DE = 10$, and $EB = 15$; and the angle $DCE$ equal to the angle $ECB$. Find the angle $DCE$, and the sides $AC$ and $BC$.

**2744. Proposed by J. B. REYNOLDS, Lehigh University.**

The vertices of a triangle are $(0, 0)$, $(2a, 0)$, and $(2x, 2y)$. Where are the vertices of the triangle of least area having its vertices on the perpendicular bisectors of the sides of the given triangle and the same center of gravity as the given triangle.

**2745. Proposed by C. N. SCHMALL, New York City.**

In the parabola, $y^2 = 4ax$, two normals to the curve are drawn at the ends of a focal chord. Show that the area between these normals and the curve is $20a^2/3 \sin^3 2\phi$, where $\phi$ is the angle between one of the normals and the $x$-axis.

**2746. Proposed by A. CAMPBELL, St. Johnsbury, Vt.**

Given the base, the sum of the sides of the triangle, and the difference of the base angles, to construct the triangle.

**2747. Proposed by ENOS W. WITMER, Sophomore in Franklin and Marshall College.**

Investigate the problem of solving the equation, $x^4 + ay^4 = w^2 + av^2$. Carmichael's *Diophantine Analysis*, problem 18, page 54.

**2748. Proposed by ROGER E. MOORE, University of Wisconsin.**

Test for convergence, the series $\sum_{n=1}^{\infty} a_n$, in which

$$a_n = \left[\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}\right]^2.$$

**2749. Proposed by A. M. HARDING, University of Arkansas.**

Through a point, $P$, within a circle, draw a chord which will be trisected at $P$.

**2750. Proposed by J. W. LASLEY, JR., University of North Carolina.**

Given $\bar{x} = \tan^{-1}\dfrac{z}{\sqrt{x^2+y^2}}$, $\bar{y} = \tan^{-1}\dfrac{y}{x}$, and $\bar{z} = \log\sqrt{x^2+y^2+z^2}$, solve for $x$, $y$, and $z$ in terms of $\bar{x}$, $\bar{y}$, and $\bar{z}$.

**2751. Proposed by J. L. RILEY, Stephenville, Texas.**

Every number whose square is the sum of the squares of two consecutive integers is equal to the sum of the squares of three integers of which two, at least, are consecutive.

**2752. Proposed by PAUL CAPRON, U. S. Naval Academy.**

Given a parallelogram with center $O$, vertices $PQP'Q'$, mid-points of sides $ABA'B'$ (cyclic order $PAQBP'A'Q'B'$). Let $K$ be any point of $OA$. Draw $KLH$ parallel to $Q'Q$ cutting $AQ$ at $L$, and draw $BLM$, meeting $OA$ produced at $M$. Draw $MH$, parallel to $PP'$, to meet $KLH$ at $H$, and draw $B'KE$ to meet $BL$ at $E$. Repeat, changing $A$, $B$, $P$, $Q$ to $A'$, $B'$, $P'$, $Q'$, respectively, and *vice versa*. Repeat each of the foregoing, changing $P$, $P'$, $B$, $B'$ to $Q$, $Q'$, $B'$, $B$, respectively, and *vice versa*.

What are the loci of $E$ and $H$? Show that $EH$ passes through $A'$ when $K$ is a point of $OA$, through $A$ when $K$ is a point of $OA'$. Consider the effect of interchanging the rôles of $A$ and $B$.

(This construction, as commonly given, is specialized in these particulars: the parallelogram is rectangular, the divisions of $OA$ are equal, and the locus of $H$ is not found.)

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## SOLUTIONS OF PROBLEMS.

**385 (Calculus). Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.**

If $f(x)$ is continuous between $a$ and $x$, show that

$$\lim_{n=\infty} \frac{\lfloor n}{(x-a)^n}\int_a^x \cdots \int_a^x f(x)dx^n = f(a).$$

### SOLUTION BY LOUIS WEISNER, Student, College of the City of New York.

Let $f(x) = \phi^{(n)}(x)$, $n$ being finite.

Hence,

$$\int_a^x f(x)dx = \phi^{(n-1)}(x) - \phi^{(n-1)}(a),$$

$$\int_a^x \int_a^x f(x)dx^2 = \phi^{(n-2)}(x) - \phi^{(n-2)}(a) - (x-a)\phi^{(n-1)}(a),$$

and

$$\int_a^x \int_a^x \int_a^x f(x)dx^3 = \phi^{(n-3)}(x) - \phi^{(n-3)}(a) - (x-a)\phi^{(n-2)}(a) - \frac{(x-a)^2}{2}\phi^{(n-1)}(a).$$

Integrating $n$ times, we have

$$\int_a \cdots \int_a f(x)dx^n = \phi(x) - \phi(a) - (x-a)\phi'(a) - \frac{(x-a)^2}{\lfloor 2}\phi''(a) - \cdots - \frac{(x-a)^{n-1}}{\lfloor n-1}\phi^{(n-1)}(a).$$

But

$$\phi(x) = \phi(a) + (x-a)\phi'(a) + \frac{(x-a)^2}{\lfloor 2}\phi''(a) + \cdots + \frac{(x-a)^{n-1}}{\lfloor n-1}\phi^{(n-1)}(a) + \frac{(x-a)^n}{\lfloor n}\phi^{(n)}(a)$$
$$+ \frac{(x-a)^{n+1}}{\lfloor n+1}\phi^{(n+1)}(a) + \frac{(x-a)^{n+2}}{\lfloor n+2}\phi^{(n+2)}(a) + \cdots.$$

Hence,

$$\int_a^x \cdots \int_a^x f(x)dx^n = \frac{(x-a)^n}{\lfloor n} \phi^{(n)}(a) + \frac{(x-a)^{n+1}}{\lfloor n+1} \phi^{(n+1)}(a) + \frac{(x-a)^{n+2}}{\lfloor n+2} \phi^{(n+2)}(a) \cdots$$

$$= \frac{(x-a)^n}{\lfloor n} f(a) + \frac{(x-a)^{n+1}}{\lfloor n+1} f'(a) + \frac{(x-a)^{n+2}}{\lfloor n+2} f''(a) + \cdots.$$

Hence,

$$\frac{\lfloor n}{(x-a)^n} \int_a^x \cdots \int_a^x f(x)dx^n = f(a) + \frac{x-a}{n+1} f'(a) + \frac{(x-a)^2}{(n+2)(n+1)} f''(a) + \cdots$$

which when $n$ increases without limit, becomes,

$$\operatorname{limit}_{n=\infty} \frac{\lfloor n}{(x-a)^n} \int_a^x \cdots \int_a^x f(x)dx^n = f(a).$$

**425 (Calculus). Proposed by O. S. ADAMS, U. S. Coast and Geodetic Survey, Washington, D. C.**

Show that the infinite product

$$(1-z)(1+\tfrac{1}{2}z)(1-\tfrac{1}{3}z)(1+\tfrac{1}{4}z)\cdots = \frac{\sqrt{\pi}}{\Gamma(1+\frac{1}{2}z)(\frac{1}{2}-\frac{1}{2}z)}.$$

SOLUTION BY C. F. GUMMER, Kingston, Ont.

The right member should evidently have been printed

$$\frac{\sqrt{\pi}}{\Gamma(1+\frac{1}{2}z)\Gamma(\frac{1}{2}-\frac{1}{2}z)},$$

since this equals

$$\frac{\Gamma(\frac{1}{2})}{\Gamma(1+\frac{1}{2}z)\Gamma(\frac{1}{2}-\frac{1}{2}z)},$$

$$= \operatorname{Lim}_{n\to\infty}\left[\left(\frac{n^{\frac{1}{2}}}{\frac{1}{2}} \prod_{i=1}^{n} \frac{i}{i+\frac{1}{2}}\right)\left(\frac{1+\frac{1}{2}z}{n^{1+\frac{1}{2}z}} \prod_{i=1}^{n} \frac{i+1+\frac{1}{2}z}{i}\right)\left(\frac{\frac{1}{2}-\frac{1}{2}z}{n^{\frac{1}{2}-\frac{1}{2}z}} \prod_{i=1}^{n} \frac{i+\frac{1}{2}-\frac{1}{2}z}{i}\right)\right]$$

$$= \operatorname{Lim}_{n\to\infty}\left[\frac{(1-z)(1+\frac{1}{2}z)}{n} \prod_{i=1}^{n}\left\{\left(1-\frac{z}{2i+1}\right)\left(1+\frac{z}{2i+2}\right)\frac{i+1}{i}\right\}\right]$$

$$= \operatorname{Lim}_{n\to\infty}\left[(1-z)(1+\tfrac{1}{2}z) \prod_{i=1}^{n}\left\{\left(1-\frac{z}{2i+1}\right)\left(1+\frac{z}{2i+2}\right)\right\}\right]$$

$$= \prod_{i=0}^{\infty}\left\{\left(1-\frac{z}{2i+1}\right)\left(1+\frac{z}{2i+2}\right)\right\},$$

as required.

**440 (Calculus). Proposed by J. B. REYNOLDS, Lehigh University.**

If $t$ is the differential vector joining two consecutive points on a space curve, and $R$ is the radius of curvature at that point prove that

$$R^2 = \frac{(t\cdot t)^3}{(t\times i)\cdot(t\times i)}.$$

I. SOLUTION BY LOUIS WEISNER, Student, College of the City of New York.

Let $r = f(\tau)$ be the vector equation of the curve, $s$ its length measured from a fixed point, and $K$ the directed curvature.
Then,

$$K = \frac{d^2r}{ds^2} = \frac{\dfrac{d}{d\tau}\left[\dfrac{\dfrac{dr}{d\tau}}{\dfrac{ds}{d\tau}}\right]}{\dfrac{ds}{d\tau}} = \frac{\dfrac{d}{d\tau}\left[\dfrac{t}{\dfrac{ds}{d\tau}}\right]}{\dfrac{ds}{d\tau}},$$

or

$$K = \frac{\frac{ds}{d\tau}\dot{t} - \frac{d^2s}{d\tau^2}t}{\left(\frac{ds}{d\tau}\right)^3}.$$

Also

$$\frac{ds}{d\tau} = (t\cdot t)^{1/2};$$

whence,

$$\frac{d^2s}{d\tau^2} = \frac{t\cdot\dot{t}}{(t\cdot t)^{1/2}}.$$

Therefore,

$$K = \frac{t\cdot\dot{t}\dot{t} - t\cdot\dot{t}t}{(t\cdot t)^2};$$

whence,

$$K\cdot K = \frac{(t\cdot t)^2(\dot{t}\cdot\dot{t}) - 2(t\cdot\dot{t})^2(t\cdot t) + (t\cdot\dot{t})^2(t\cdot t)}{(t\cdot t)^4} = \frac{(t\cdot t)^2(\dot{t}\cdot\dot{t}) - (t\cdot\dot{t})^2(t\cdot t)}{(t\cdot t)^4} = \frac{t\cdot t\dot{t}\cdot\dot{t} - (t\cdot\dot{t})^2}{(t\cdot t)^3}.$$

But

$$(t\times\dot{t})\cdot(t\times\dot{t}) = t\cdot t\dot{t}\cdot\dot{t} - (t\cdot\dot{t})^2.$$

Therefore,

$$K\cdot K = \frac{(t\times\dot{t})\cdot(t\times\dot{t})}{(t\cdot t)^3}.$$

Since $R$ is defined as the reciprocal of the magnitude of $K$, it follows that

$$R^2 = \frac{1}{K\cdot K}.$$

Hence,

$$R^2 = \frac{(t\cdot t)^3}{(t\times\dot{t})\cdot(t\times\dot{t})}.$$

## II. Solution by the Proposer.

Let $P$, $P'$, $P''$ be three points on a space curve whose vector equation is

$$r = af(u)i + b\varphi(u)j + c\psi(u)k \tag{1}$$

$i$, $j$, $k$ being three mutually perpendicular unit vectors and $P$, $P'$, $P''$ located by the vectors $r$, $r'$, and $r''$ so that

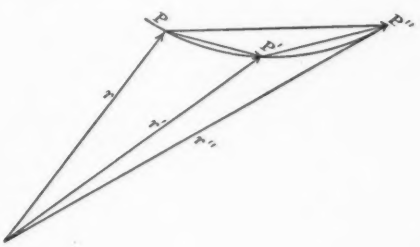
$$\left.\begin{aligned} r &= af(u)i + b\varphi(u)j + c\psi(u)k \\ r' &= af(u+h)i + b\varphi(u+h)j + c\psi(u+h)k \\ r'' &= af(u+2h)i + b\varphi(u+2h)j + c\psi(u+2h)k \end{aligned}\right\} \tag{2}$$

By trigonometry, the radius of the circle circumscribing the triangle $PP'P''$ is

$$\frac{(PP')(PP'')(P'P'')}{4A},$$

$A$ being the area of the triangle; whence, the radius of curvature of the curve, $R$, will be given by

$$R^2 = \frac{(t\cdot t)(t'\cdot t')(t''\cdot t'')}{2(t'\times t)\cdot(t'\times t)} \tag{3}$$



n which

$$\left.\begin{aligned} t &= r' - r_{h\doteq 0} \\ t' &= r'' - r'_{h\doteq 0} \\ t'' &= t' + t \end{aligned}\right\} \tag{4}$$

Expanding (2) to the second powers of $h$, we find

$$t = ha\left\{f'(u) + \frac{h}{2}f''(u)\right\}i + bh\left\{\varphi'(u) + \frac{h}{2}\varphi''(u)\right\}j + ch\left\{\psi'(u) + \frac{h}{2}\psi''(u)\right\}k_{h\doteq 0},$$

$$t' = ha\{f'(u) + \tfrac{1}{2}hf''(u)\}i + bh\{\varphi'(u) + \tfrac{1}{2}h\varphi''(u)\}j + ch\left\{\psi'(u) + \frac{3h}{2}\psi''(u)\right\}k_{h\doteq 0},$$

$$t'' = 2ha\{f(u) + hf''(u)\}i + 2bh\{\varphi'(u) + h\varphi''(u)\}j + 2ch\{\psi'(u) + h\psi''(u)\}k_{h\doteq 0}.$$

$$t \times t = h^2ab\{f'(u)\varphi''(u) - f''(u)\varphi'(u)\}i + h^2bc\{\varphi'(u)\psi''(u) - \varphi''(u)\psi'(u)\}j$$
$$+ h^2ac\{\psi'(u)f''(u) - \psi''(u)f'(u)\}k_{h\doteq 0}.$$

Substituting these values in (3), we find as $h = 0$

$$R^2 = \frac{(t\cdot t)^3}{(t \times t)\cdot(t \times t)}.$$

In the case of the helix,

$$r = a\cos ui + a\sin uj + cuk; \qquad t\cdot t = (a^2 + c^2)h^2{}_{h\doteq 0};$$

and

$$t \times t = h^2\{a^2i + ac\sin uj - ac\cos uk\}_{h\doteq 0}.$$

Hence,

$$R = \frac{a^2 + c^2}{a}.$$

NOTE.—The dots over the $t$'s in the denominator of the problem as it appeared in the MONTHLY are misprints, according to a statement of the Proposer. We are publishing Mr. Weisner's solution of the problem as it appeared in print and also Professor Reynolds's solution of the problem as he wished it to appear.—EDITOR OF PROBLEMS AND SOLUTIONS.

**322 (Mechanics). Proposed by FRANK R. MORRIS, Glendale, California.**

A pole of uniform size and weight throughout its length stands in a vertical position. The height of the pole is $h$ and its weight $w$. It hinges at the base and falls, passing through a horizontal position. At the moment it reaches the horizontal position, how far from the base is the maximum vertical force tending to break the pole? How great is this force? What is the horizontal force at the same position in the pole?

## I. SOLUTION BY C. F. GUMMER, Kingston, Ontario.

Let the rod coincide initially with the $y$-axis and later with the $x$-axis. Let $\theta$ be its inclination to the latter at time $t$; $P$, $Q$, and $N$ the radial and transverse stresses and the breaking couple at distance $r$ from the origin.

The equations of motion for the element $dr$ are

(1) $$\frac{\partial P}{\partial r}\cos\theta - \frac{\partial Q}{\partial r}\sin\theta = \frac{wr}{gh}\left(\frac{d}{dt}\right)^2\cos\theta,$$

(2) $$\frac{\partial P}{\partial r}\sin\theta + \frac{\partial Q}{\partial r}\cos\theta = \frac{wr}{gh}\left(\frac{d}{dt}\right)^2\sin\theta + \frac{w}{h},$$

and

(3) $$Q + \frac{\partial N}{\partial r} = 0.$$

The boundary conditions are

(4) $$P = Q = N = 0 \quad \text{when} \quad r = h, \text{ and}$$

(5) $$N = 0 \quad \text{when} \quad r = 0.$$

The initial condition is

6) $$\frac{d\theta}{dt} = 0 \quad \text{when} \quad \theta = \frac{\pi}{2}.$$

By integrating (1) and (2) for $r$, and using (4), we get

$$P = \frac{w}{2gh}(h^2 - r^2)\left(\frac{d\theta}{dt}\right)^2 - \frac{w}{h}(h - r)\sin\theta,$$

$$Q = -\frac{w}{2gh}(h^2 - r^2)\frac{d^2\theta}{dt^2} - \frac{w}{h}(h - r)\cos\theta,$$

and

$$N = -\int_0^r Q dr = \frac{w}{2gh}\left(h^2 r - \frac{r^3}{3}\right)\frac{d^2\theta}{dt^2} + \frac{w}{h}\left(hr - \frac{r^2}{2}\right)\cos\theta.$$

Hence, by (5),

$$2h\frac{d^2\theta}{dt^2} = -3g\cos\theta$$

and hence, by (6),

$$h\left(\frac{d\theta}{dt}\right)^2 = 3g(1 - \sin\theta).$$

It follows that, when $\theta = 0$,

$$P = \frac{3w}{2h^2}(h^2 - r^2), \qquad Q = \frac{w}{4h^2}(h - r)(3r - h), \qquad \text{and} \qquad N = \frac{w}{4h^3}r(h - r)^2.$$

Therefore, the shear, $Q$, is a maximum for $r = 2h/3$, which makes $P = 5w/6$, $Q = w/12$, and $N = hw/54$.

It may be added that $N$ is a maximum for $r = h/3$, which gives $P = 4w/3$, $Q = 0$, and $N = hw/27$.

This problem is discussed in E. J. Routh's *Elementary Rigid Dynamics*, § 151, 7th edition.

## II. Solution by J. B. Reynolds, Lehigh University, South Bethlehem, Pa.

At any instant, let the pole make an angle $\theta$ with the vertical and let $O$ be the point about which it is turning. Taking moments about $O$, we have

$$w\frac{h}{2}\sin\theta = \frac{w}{g}\frac{h^2}{3}\ddot{\theta},$$

or

$$\omega\frac{d\omega}{d\theta} = \ddot{\theta} = \frac{3g}{2h}\sin\theta, \tag{1}$$

whence

$$\omega^2 = \dot{\theta}^2 = \frac{3g}{h}(1 - \cos\theta), \tag{2}$$

$\omega$ being the angular velocity of the pole.

When in a horizontal position, let any differential mass of the pole of mass $k$ per unit length be acted upon by a vertical stress force $F$ and a horizontal force $T$. If this differential of mass is at distance $r$ from $O$, its accelerations will be $r\ddot{\theta}$ vertically and $r\dot{\theta}^2$ horizontally.

Resolving vertically, we have

$$F + gdm = r\ddot{\theta}dm, \tag{3}$$

in which $dm = kdr$.

Then if $M$ is the moment of the couple tending to break the pole at a point distant $u$ from $O$, we have

$$M = kg\int_u^h F(r - u) = \int_u^h (r\ddot{\theta} - g)(r - u)dm, \text{ by (3), or by (1) for } \theta = 90°,$$

$$M = kg\int_u^h (3r/2h - 1)(r - u)dr = w/2h^2\int_u^h (3r - 2h)(r - u)dr, \ = w/4h^2\{h^2u + u^3 - 2hu^2\}$$

since $kgh = w$.

For this to be a maximum $dM/du = 0$, or $3u^2 - 4hu + h^2 = (3u - h)(u - h) = 0$; whence $u = 1/3h$, giving the point at which there is the greatest tendency to break. At this point, $M = wh/27$.

Resolving horizontally for a differential mass, we have $T + dT - T = r\dot{\theta}^2 dm$, so that by (2) for $\theta = 90°$,

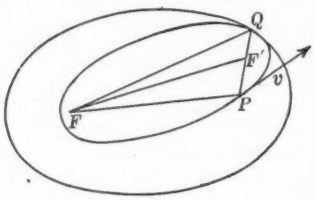
$$T = \frac{3gk}{h}\int_{1/3h}^h rdr = \frac{3w}{h^2}\cdot\frac{r^2}{2}\Big]_{1/3h}^h = \tfrac{4}{9}w.$$

**328 (Mechanics). Proposed by J. B. REYNOLDS, Lehigh University.**

Find the envelope of all possible trajectories when a particle is projected with a constant velocity, $v$, from a fixed point at a distance $a$ from the center of attraction under the law of gravitation.

## I. Solution by the Proposer.

For parabolic or hyperbolic velocities there is evidently no envelope for all directions of projection as any point in space may be reached. For velocities giving elliptic orbits we have




$$v^2 = \mu\left(\frac{2}{a} - \frac{1}{A}\right) \tag{1}$$

in which $A$ is the semi-major axis of the orbit; whence

$$A = \frac{\mu a}{2\mu - av^2} \tag{2}$$

In the figure, let $P$ be the point of projection, $F$ and $F'$ the foci of the path and $Q$ the point where $PF'$ extended cuts the orbit. We will prove that the locus of $Q$ is an ellipse.

Now

$$\begin{aligned} FQ + F'Q &= 2A \\ a \quad + F'P &= 2A. \end{aligned} \tag{1}$$

Adding, we get

$$FQ + F'Q + F'P = 4A - a = \frac{2\mu a + v^2a^2}{2\mu - av^2} = a \cdot \frac{2\mu + av^2}{2\mu - av^2}. \tag{2}$$

or

$$FQ + PQ = \text{const.} \tag{3}$$

Hence, the locus of $Q$ is an ellipse. Furthermore, at $Q$ both ellipses have the same normal so that the inner ellipse does not cross the outer. The ellipsoid of revolution generated by revolving the outer ellipse about $FP$ is the envelope of all the trajectories in question.

If the major axis of the generating ellipse is $2a'$ and its eccentricity $e'$, we have by (2)

$$2a' = a\frac{2\mu + av^2}{2\mu - av^2}. \tag{4}$$

Suppose the particle projected directly away from $F$. Then since $vdv/dr = -\mu/r^2$, we have, when the particle comes to rest at distance $r$ from $F$,

$$-v^2 = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right) \quad \text{or} \quad r = \frac{2\mu a}{2\mu - av^2}.$$

This value of $r$ is evidently $a'(1 + e')$; so that

$$a'(1 + e') = \frac{2\mu a'}{2\mu - av^2} \cdots. \tag{5}$$

Now (4) and (5) give

$$e' = \frac{2\mu - av^2}{2\mu + av^2},$$

and the envelope is consequently determined.

## II. Solution by C. F. Gummer, Kingston, Ontario.

The origin being the center of attraction, $(a, 0)$ the initial position, and $\mu/\rho^2$ the attraction, the equations of motion will be

(1) $$\ddot{\rho} - \rho\dot{\theta}^2 = -\mu\rho^{-2},$$

and

(2) $$2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} = 0.$$

By (2), $\rho^2\dot{\theta} = A$, so that (1) gives

$$\ddot{\rho} - A^2\rho^{-3} = -\mu\rho^{-2},$$

or

$$\dot{\rho} = \sqrt{-A^2\rho^{-2} + 2\mu\rho^{-1} + B}.$$

The speed is

$$\sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2} = \sqrt{2\mu\rho^{-1} + B}.$$

Hence,

$$B = v^2 - 2\mu a^{-1}.$$

Hence,

$$\frac{d\rho}{d\theta} = \frac{\dot{\rho}}{\dot{\theta}} = \rho^2 A^{-1}\sqrt{v^2 - 2\mu a^{-1} + 2\mu\rho^{-1} - A^2\rho^{-2}}$$

and therefore,

$$\theta = -\int_{a^{-1}}^{\rho^{-1}} A^{-1}\sqrt{v^2 - 2\mu a^{-1} + 2\mu z - A^2 z^2}\, dz;$$

that is,

$$\rho^{-1} = \mu A^{-2} - (\mu A^{-2} - a^{-1})\cos\theta + \sqrt{v^2 A^{-2} - a^{-2}}\cdot\sin\theta,$$

which is the equation of the trajectory.

Writing this in the form,

$$\mu^2 a^2(1-\cos\theta)^2 + \{2\mu a(1-\cos\theta)(\cos\theta - a\rho^{-1}) - a^2v^2\sin^2\theta\}A^2 + \{(\cos\theta - a\rho^{-1})^2 + \sin^2\theta\}A^4 = 0,$$

and applying the condition for equal roots in $A^2$, we get for the envelope

$$\frac{4\mu a^2v^2}{4\mu^2 - a^2v^4}\frac{1}{\rho} = 1 - \frac{2\mu - av^2}{2\mu + av^2}\cos\theta,$$

the equation of an ellipse.

**349 (Mechanics). Proposed by S. A. COREY, Albia, Iowa.**

A 9-pound weight is attached to a string which passes over a smooth fixed pulley. The other end of the string is fastened to and supports a smooth pulley $P_1$ of weight 1 pound over which passes a second string, one end attached to a 3-pound weight and the other end attached to and supporting another smooth pulley $P_2$ of weight 1 pound. Over the pulley $P_2$ passes a third string supporting weights 2 pounds and $3\frac{1}{3}$ pounds.

If the system is acted upon by gravity alone show that the acceleration of the 9-pound weight, $3\frac{1}{3}$-pound weight, and pulley $P_2$ are 0, $\frac{1}{7}g$, and $\frac{1}{14}g$, respectively.

Determine the motion of the weights when pulleys are not smooth, that is, when friction is present.

## II. Solution by William Hoover, Columbus, Ohio.

Calling the fixed pulley $P$ and taking it as the origin of coördinates, we have for the $x$-coordinate of $m_1$, $x_1$; of $P_1$, $l_1 - x_1$; of $m_3$, $x_2 + l_1 - x_1$; of $P_2$, $l_1 + l_2 - (x_1 + x_2)$; of $m_4$, $x_3 + l_1 + l_2 - (x_1 + x_2)$; of $m_5$, $l_1 + l_2 + l_3 - (x_1 + x_2 + x_3)$, where $m_1$, $m_3$, $m_4$ and $m_5$ are the masses of the 9-, 3-, 2-, and $3\frac{1}{3}$-pound weights, respectively.

Under the hypothesis that there is no friction, the equation of motion of system is

$$T = \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}P_1\dot{x}_1^2 + \tfrac{1}{2}m_3(\dot{x}_2 - \dot{x}_1)^2 + \tfrac{1}{2}(\dot{x}_1 + \dot{x}_2)^2 + \tfrac{1}{2}m_4(\dot{x}_3 - \dot{x}_1 - \dot{x}_2)^2 + \tfrac{1}{2}m_5(\dot{x}_1 + \dot{x}_2 + \dot{x}_3)^2 = m_1gx_1 + P_1g(l_1 - x_1) + m_3g(x_2 + l_1 - x_1) + P_2g\{l_1 + l_2 - (x_1 + x_2)\} + m_4g\{l_1 + l_2 + x_3 - (x_1 + x_2)\} + C = V. \qquad (i)$$

Using Lagrange's equations of type

$$\frac{d}{dt}\frac{dT}{d\dot{x}} - \frac{dT}{dx} = \frac{dV}{dx} \qquad (ii)$$

there are

(a) $\quad \frac{8}{3}\ddot{x}_1 + \frac{1}{3}\ddot{x}_2 + \frac{1}{3}\ddot{x}_3 = g(m_1 - P_1 - m_3 - P_2 - m_4 - m_5) = -\frac{1}{3}g,$

(b) $\quad \frac{1}{3}\ddot{x}_1 + \frac{2}{3}\ddot{x}_2 + \frac{1}{3}\ddot{x}_3 = g(\quad m_3 - P_2 - m_4 - m_5) = -\frac{1}{3}g,$

(c) $\quad \frac{1}{3}\ddot{x}_1 + \frac{1}{3}\ddot{x}_2 + \frac{1}{3}\ddot{x}_3 = g(\quad m_4 - m_5) = -\frac{1}{7}g,$

giving $\ddot{x}_1 = 0$, $\ddot{x}_2 = -\frac{g}{3}$, $\ddot{x}_3 = -\frac{g}{6}$.

[In the previously printed solution nothing is given to show *how* the *literal* terms on right of (*a*), (*b*), (*c*) arise.]

In case of friction of the movable pulleys, account generally would have to be taken of the *vis viva* of rotary motion of $P_1$ and $P_2$, there being, then, two additional terms in the energy equation $T = V$, namely,

$$\frac{k_1^2}{2} P_1 \dot{\varphi}_1^2 \quad \text{and} \quad \frac{k_2^2}{2} P_2 \dot{\varphi}_2^2,$$

$k_1$ and $k_2$ being the radii of gyration of the pulleys $P_1$, $P_2$, supposed to be cylinders of radii, say, $r_1$, $r_2$, and so $k_1^2 = r_1^2/2$, $k_2^2 = r_2^2/2$, and $\dot{\varphi}_1$, $\dot{\varphi}_2$, the angular velocities of $P_1$, $P_2$.

There were not stated any dimensions of $P_1$, $P_2$; permitting us to take $r_1$ and $r_2$ indefinitely small. This would, though, be a special condition.

In taking $r_1$, $r_2$ of finite value, the energy equation would be

$$T = \tfrac{1}{2}(m_1 + P_1 + m_3 + P_2 + m_4 + m_5)\dot{x}_1^2 + \text{etc.} + (-m_3 + P_2 + m_4 + m_5)\dot{x}_1\dot{x}_2 + \text{etc.}$$
$$+ \tfrac{1}{2}k_1^2\dot{\varphi}_1^2 + \tfrac{1}{2}k_2^2\dot{\varphi}_2^2 = m_1 g x_1 + \text{etc.} + C = V. \quad (d)$$

Regardless of signs, $r_1\dot{\varphi}_1 = \dot{x}_1$, $r_2\varphi_2 = x_2$; and substituting $\dot{\varphi}_1$, $\dot{\varphi}_2$ from these equations in (*d*), we have an equation differing in form from (*i*), but to which (*ii*) may be applied as before, and generally with different values for $\ddot{x}_1$ etc.

So far the fixed pulley has been thought of as smooth. If this pulley be regarded as rough, the energy equation would still further be modified so as to take account of work done against friction, or of energy due to its rotation and some assigned mass

**350 (Mechanics). Proposed by J. B. REYNOLDS, Lehigh University.**

If an elastic tube filled with liquid under pressure doubles in length in what ratio will the radius be increased?

## SOLUTION BY THE PROPOSER.

Let the initial length and radius of the tube be $l_0$ and $r_0$, the final lengths $l$ and $r$ and the coefficient of elasticity of a strip of the material one unit wide be $\lambda$.

If $P$ is the pressure per square unit we have, since the length is doubled,

$$P\pi r^2 = 2\pi r \frac{\lambda}{l_0} l_0; \qquad \text{whence} \qquad P = \frac{2\lambda}{r}. \quad (1)$$

Again, if $T$ is the peripheral tension per unit length, in order to balance the internal pressure

$$T = \lim_{\Delta\theta \doteq 0} \frac{Pr\Delta\theta}{2 \sin \Delta\theta/2} = Pr. \quad (2)$$

Also,

$$T = \frac{\lambda}{2\pi r_0}(2\pi r - 2\pi r_0) = \frac{\lambda}{r_0}(r - r_0). \quad (3)$$

By (1), (2), and (3) we get,

$$Pr = 2\lambda = \frac{\lambda}{r_0}(r - r_0);$$

whence $r = 3r_0$.

**356 (Mechanics). Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore, Md.**

A ray of light enters a prism having vertex angle $\alpha$. If the angle between the incoming and outgoing directions is defined as the angle of deviation, at what angle must the ray enter the prism in order that the angle of deviation may be a minimum?

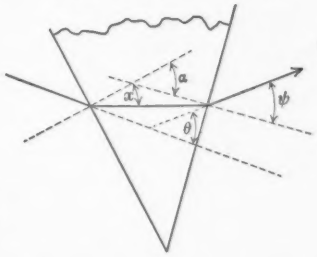
## SOLUTION BY J. B. REYNOLDS, Lehigh University.

We have

$$\sin\varphi = \epsilon \sin x, \qquad \sin\psi = \epsilon \sin(\alpha - x),$$

$\epsilon$ being the index of refraction. If $\theta$ is the angle of deviation, we have, by geometry, $\theta = \psi + \varphi - \alpha$ or

$$\theta = \sin^{-1}\epsilon \sin(a - x) + \sin^{-1}\epsilon \sin x - \alpha.$$




For $\theta$ to be a minimum

$$\frac{d\theta}{dx} = \frac{-\epsilon\cos(\alpha - x)}{\sqrt{1 - \epsilon^2\sin^2(\alpha - x)}} + \frac{\epsilon\cos x}{\sqrt{1 - \epsilon^2\sin^2 x}} = 0;$$

from which $x = \frac{1}{2}\alpha$ and therefore $\varphi = \psi = \sin^{-1}\epsilon \sin\frac{1}{2}\alpha$. That is, the angle of deflection or deviation is a minimum when the ray in the glass makes equal angles with the faces of the prism.

**357 (Mechanics). Proposed by J. B. REYNOLDS, Lehigh University, South Bethlehem, Pa.**

Two beads each of mass $m$ connected by a string of length $2l$ and carrying a mass $m'$ at its middle point are threaded symmetrically with respect to the major axis which is vertical on a smooth ellipse of eccentricity $e$ and latus rectum $2l$. The string is held taut and horizontal, then released; find the velocities of the beads when the end ones impinge.

## Solution by the Proposer.

There are two cases: I, when the end beads are at the extremities of the upper latus rectum II, when the end beads are at the extremities of the lower latus rectum. In either case when the end beads impinge their velocities will be equal and the velocity of $m'$ will be zero. If $a$ is the semi-major axis we have by the principle of work,

Case I

$$m'g\{l - a(1 - e)\} - 2mga(1 - e) = \frac{2m}{2}v^2;$$

or since

$$l = a(1 - e^2),$$

$$mv^2 = ag(1 - e)(m'e - 2m);$$

or

$$v^2 = \frac{gl}{m(1 + e)}\{m'e - 2m\}.$$

For the beads to impinge in this case $m' > 2m/e$.

Case II

$$m'g\{l + a(1 - e)\} + 2mga(1 - e) = \frac{2m}{2}v^2,$$

whence as before

$$v^2 = \frac{gl}{m(1 + e)}\{(2 + e)m' + 2m\}.$$

**260 (Number Theory). Proposed by ALBERT A. BENNETT, University of Texas.**

Let $\binom{n}{r}$ denote, as usual, the binomial coefficient $n!/[r!(n - r)!]$, where $\binom{n}{0} = 1$, but where $n$, $r$, $(n - r)$ are always to be supposed to be positive integers or zero. Let us define $k_i(m, n)$ as $\Sigma_i \binom{m-i+j}{i-j}\binom{n-j}{j}$. Prove that the following recursion formula is consistent:

$$\Sigma(-1)^i k_i(m, n)C_{m+n-i} = \binom{m+n}{m}$$

and determine $C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, C_6 = 132, C_7 = 429, C_8 = 1{,}430$, etc. Prove also that these quantities satisfy the following relations, as well:

$$\sum_i (-1)^i C_{m-n-i}\binom{m-1}{i} = 0$$

for each $n$ where $2n \leq m$.

## Solution by C. F. Gummer, Kingston, Ontario.

Consider the sum $\Sigma_i \Sigma_m \Sigma_n (-1)^i k_i(m,n) x^m y^n z^i$, where $x, y, z$ are small enough to ensure absolute convergence. It equals $\Sigma_i \Sigma_j \Sigma_m \Sigma_n (-1)^i \binom{m-i+j}{i-j}\binom{n-j}{j} x^m y^n z^i$. It is not difficult to sum this in the order of writing, beginning with $n$. The result is that $(-1)^i k_i(m,n)$ is the coefficient of $x^m y^n z^i$ in $(1-x+x^2z)^{-1}(1-y+y^2z)^{-1}$, as may also be directly verified. It will then be sufficient to show that a power series $F(z) = C_0 + C_1 z + C_2 z^2 + \cdots$, independent of $x$ and $y$ can be found so that the coefficient of $x^m y^n z^{m+n}$ in $F(z)\cdot(1-x+x^2z)^{-1}(1-y+y^2z)^{-1}$ equals the corresponding coefficient in $\Sigma_m \Sigma_n \binom{m+n}{m} x^m y^n z^{m+n}$, that is, in $(1-xz-yz)^{-1}$. This does not mean that these generating functions are to be identical, since the first contains terms not of the form $Ax^m y^n z^{m+n}$. To escape this difficulty, write $x = u/z$, $y = v/z$. Then we have to find $F(z)$ so that the part independent of $z$ in $F(z)\cdot(1-u/z+u^2/z)^{-1}(1-v/z+v^2/z)^{-1}$ is identical with $(1-u-v)^{-1}$, remembering that we are now to have positive and negative powers of $z$ in the expansion of the former.

But

$$(1-u/z+u^2/z)^{-1}(1-v/z+v^2/z)^{-1} = (u-v)^{-1}(1-u-v)^{-1}\{(1-u/z+u^2/z)^{-1} - (1-v/z+v^2/z)^{-1}\};$$

so that, on expanding this in negative powers of $z$, and multiplying by $F(z)$, we find that $\Sigma_r C_r\{(u-u^2)^{r+1} - (v-v^2)^{r+1}\}$ must be identical with $u-v$. This can be satisfied only by having

$$\sum_r C_r(u-u^2)^{r+1} = u + A,$$

that is by making $zF(z) - A$ the expansion of $u$ in ascending powers of $z$ where $u^2 - u + z = 0$. Therefore,

$$zF(z) = A + \tfrac{1}{2} \pm \tfrac{1}{2}(1-4z)^{1/2};$$

whence,

$$A = -\tfrac{1}{2} \mp \tfrac{1}{2},$$

and

$$F(z) = \pm\{1-(1-4z)^{1/2}\}/(2z);$$

and the upper sign must be taken, since that alone makes $u + v < 1$, which is necessary for the convergence of $(1-u-v)^{-1}$. It follows that $C_r = 2(2r-1)!/[(r-1)!(r+1)!]$ when $r > 0$, and $C_0 = 1$. The numerical values may then be found, as stated.

As regards the second part of the problem, we observe that $\Sigma_i (-1)^i \binom{m-i}{i} C_{m-n-i}$ is the coefficient of $x^m z^{m-n}$ in the expansion of $F(z)\cdot\{1-(x-zx^2)\}^{-1}$ for small $x$ and $z$; and by using partial fractions in terms of $x$, we find that this coefficient is the coefficient of $z^{m-n}$ in

$$\{1-(1-4z)^{1/2}\}(2z)^{-1}\cdot[\{\tfrac{1}{2}+\tfrac{1}{2}(1-4z)^{1/2}\}^{m+1} - \{\tfrac{1}{2}-\tfrac{1}{2}(1-4z)^{1/2}\}^{m+1}](1-4z)^{-1/2}.$$

The second of the terms in square brackets has no powers of $z$ below $z^{m+1}$, and may be omitted. The remaining part reduces to $\{\frac{1}{2}+\frac{1}{2}(1-4z)^{1/2}\}^m(1-4z)^{-1/2}$. For a similar reason, for powers of $z$ below $z^m$, this may be replaced by

$$[\{\tfrac{1}{2}+\tfrac{1}{2}(1-4z)^{1/2}\}^m - \{\tfrac{1}{2}-\tfrac{1}{2}(1-4z)^{1/2}\}^m](1-4z)^{-1/2} = 2^{-m+1}\left\{\binom{m}{1} + \binom{m}{3}(1-4z) + \binom{m}{5}(1-4z)^2 + \cdots\right\}$$

which has a degree less than $m/2$. The coefficient considered therefore vanishes when $0 < 2n \leq m$. It is equal to 1 when $n = 0$, since

$$\sum_i (-1)^i \binom{m-i}{i} C_{m-i} = \sum_i (-1)^i \binom{m-1-i}{i} C_{m-1-i} + \sum_i (-1)^i \binom{m+1-i}{i} C_{m-i}$$

$$= \sum_i (-1)^i \binom{m-1-i}{i} C_{m-1-i} = \sum_i (-1)^i \binom{m-2-i}{i} C_{m-2-i} = \cdots$$

$$= \sum_i (-1)^i \binom{1-i}{i} C_{1-i} = 1.$$

# NOTES AND NEWS

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Illinois.

The editor will appreciate the coöperation of readers in reporting news to him concerning teachers of collegiate mathematics and others interested in advanced mathematics.

Dr. H. A. KINGSTON has been promoted to an assistant professorship of mathematics and astronomy at the University of Manitoba.

Professor H. M. SHOWMAN, of the Colorado School of Mines, is studying at Harvard University as Shattuck Scholar, holding also an appointment as lecturer on physics.

Assistant Professor L. A. H. WARREN has been appointed acting professor of mathematics and astronomy at the University of Manitoba, in the absence of Professor N. B. MACLEAN, who is major in the Canadian artillery service in France.

Professor CLYDE S. ATCHISON, of Washington and Jefferson College, has returned to his college work after spending the summer as cost accountant and auditor in the service of the U. S. Shipping Board at Bath, Maine.

Dr. ARTEMAS MARTIN, computer in the Division of Tides in the U. S. Coast and Geodetic Survey, a charter member of the Association, and well known mathematician, died in Washington City on November 7, in the eighty-fourth year of his life. Until fifty years of age, he devoted his time to wood-chopping, oil-well drilling, farming and teaching (four winters), his leisure time being spent in mathematical study. In 1878 he began the publication of the *Mathematical Visitor*, of which the last number, volume 2, number 4, dated January, 1894, appeared in October, 1895. This periodical was followed, in 1882, by the *Mathematical Magazine* which continued at irregular intervals to volume 2, number 12, part 2, September, 1910. Dr. Martin was not a college graduate but his mathematical attainments secured for him honorary academic distinction. Yale conferred an honorary A.M. upon him in 1877, Rutgers granted him the Ph.D. in 1882, and Hillsdale College the LL.D. in 1885. Dr. Martin possessed one of the finest private collections of mathematical books in America; this collection was given to the American University, shortly before his death, to found the Artemas Martin Library. He also endowed the Artemas Martin Lectureship in mathematics and physics at the same university.

Lieutenant W. L. HART of the Coast Artillery has been advanced to the rank of major and is serving in France.

Dr. C. H. YEATON has been appointed instructor in mathematics at the University of Minnesota.

Dr. M. G. GABA, of Cornell University, has been appointed associate professor of mathematics at the University of Nebraska.

At the University of Saskatchewan, Assistant Professor L. L. DINES has been promoted to a full professorship of mathematics.

Lieutenant T. R. HOLLCROFT has been released from the teaching staff of the school of artillery at Camp Taylor, and has resumed the duties of his professorship of mathematics at Wells College. During his absence the position was filled by Mrs. Hollcroft.

At the University of Missouri, Mr. F. DUNCAN, Miss Z. FERGUSON, and Mr. A. GROSSMAN have been appointed instructors in mathematics.

Professor E. R. HEDRICK, of the University of Missouri, has recently sailed for France, where he is to take charge of mathematical work for American soldiers in the Y. M. C. A. educational system.

At the University of Illinois, Dr. ELIZABETH B. GRENNAN and Dr. JOSEPHINE GLASGOW have been appointed instructors in mathematics and Mr. L. E. YAEGER assistant in mathematics. Mr. C. H. RICHARDSON, assistant in mathematics, has been appointed professor of mathematics in Georgetown College, Kentucky.

At Cornell University, Mrs. D. NAYLOR, Mrs. HELEN B. OWENS, and Mr. H. L. SMITH have been appointed instructors in mathematics. Instructor E. P. FRALEIGH resigned to engage in the ordnance work at the Aberdeen Proving Ground. Seven members of the faculty were temporarily transferred from other departments to assist in the mathematical instruction last autumn.

Mr. H. W. POWELL, tutor in mathematics in the College of the City of New York, died July 23, 1918.

Professor H. G. KEPPEL, head of the department of mathematics in the University of Florida, died October 5, at the age of fifty-two years. Dr. T. M. SIMPSON, of the University of Wisconsin, has been appointed his successor as professor and head of the department.

Col. E. W. BASS, professor of mathematics at West Point from 1878 to 1898, died November 6 at the age of seventy-six years.

Dr. R. M. WINGER, of the University of Oregon, has been appointed professor of mathematics at the University of Washington.

Dr. A. H. NORTON, professor of mathematics at Elmira College, has returned from a year's work in the Y. M. C. A. with the American Army in France.

In *The Scientific Monthly* we read of the death of the astronomer A. N. SKINNER, professor of mathematics in the U. S. Navy from 1898 to the time of his retirement from active service, with rank of Commander U. S. N., in 1907.

Professor C. CARATHÉODORY, of the University of Göttingen, has been appointed professor of mathematics at the University of Berlin, as successor to the late Professor G. FROBENIUS.

The Rev. Dr. G. M. SEARLE, superior general of the Paulist Fathers from 1904 to 1909, and previously professor of mathematics and director of the astronomical observatory of the Catholic University at Washington, died on July 8, 1918 at the age of 79 years.

Mr. ARTHUR RAMSEY, of the department of mathematics in Grove City College has been promoted to an assistant professorship.

Mr. J. W. BALDWIN, who has been an instructor in mathematics in Michigan State Normal College, has been appointed to a like position in Detroit Junior College.

Professor OTTO HÖLDER has been chosen rector of the University of Leipsic.

Mr. HERMAN BETZ, formerly of the University of Michigan, instructed S. A. T. C. classes at Hobart College, after having served in the ordnance department of the army.

Professor H. L. SLOBIN, of the University of Minnesota, was engaged in Jewish welfare work in the various army camps of the United States.

Dr. GUY W. SMITH, of Beloit College, has been appointed instructor in mathematics at the University of Kentucky.

Professor OTTO STAUDE, of the University of Rostock, has been chosen rector of the University for the year 1918–19.

Professor F. B. WILLIAMS has returned to his work in Clark University after a year spent in Y. M. C. A. work in France.

Among a party of college and university men taking the officers' course in coast artillery at Fort Monroe the past summer were the following members of the Association: Professor J. J. LUCK, of the University of Virginia, H. B. PHILLIPS, of the Massachusetts Institute of Technology and D. T. WILSON, of Case School of Applied Science.

Professor D. E. SMITH, of Columbia University, was one of the two principal speakers at the annual meeting of the Association of Teachers of Mathematics

in New England, held in Boston, December 7, 1918. At this meeting Professor W. R. RANSOM, of Tufts College, was elected President of the Association. On February 7–8, Professor Smith addressed the Association of Teachers of Secondary Mathematics of North Carolina in Greensboro, N. C. He is to speak before the Rhode Island Circle of Mathematics Teachers at Providence, R. I., March 15.

At the recent election Professor D. A. ROTHROCK, of Indiana University, was elected a member of the State Legislature. He is relieved of duties at the University during the legislative session beginning in January.

The following deaths are announced: Professor E. R. NEOVIUS, of the University of Helsingfors, on September 26, 1917, at the age of sixty-seven years; Professor J. H. GRAF, of the University of Bern, at the age of sixty-six years; Professor C. STEPHANOS, of the University of Athens, at the age of sixty years; and Dr. O. REICHEL, professor of mathematics at the Landwirtschaftliche Hochschule, Berlin, at the age of eighty-three years.

Professor MAX NOETHER, of the University of Erlangen, who is now seventy-five years of age, was relieved from lecturing after April 1, 1918.

Professor A. v. BRILL, of the University of Tübingen is also about to retire after fifty-one years of teaching.

The University of Frankfort has recently conferred the degree of honorary doctor on Professor L. KÖNIGSBERGER, of the University of Heidelberg, who was characterized as "the highly deserving and many sided investigator, the brilliant and model teacher, the intelligent interpreter of the scientific life-work of Helmholtz and Jacobi."

Dr. G. M. GREEN, instructor in mathematics at Harvard University, and one of the most promising young geometers of the country, died at Cambridge on January 25, in the twenty-ninth year of his age. He graduated from College of the City of New York, B.Sc. 1911, Columbia University, M.A. 1912, Ph.D. 1913. His thesis was entitled *Projective differential geometry of triple systems of surfaces* (Lancaster, Pa., 1913, 28 pp.). His mathematical papers appeared in: *Transactions of the American Mathematical Society*, 1914–17; *Bulletin of the American Mathematical Society*, 1917–18; *Annals of Mathematics*, 1918; and *Proceedings of the National Academy of Sciences*, 1915–18.

At Wesleyan University C. L. STEARNS has been appointed instructor in mathematics and assistant in the Van Vleck observatory.

Professor E. G. BILL, on leave of absence from Dartmouth College, has recently been appointed Assistant to Director of Military Service Branch, Department of Justice, Ottawa.

Major WILLIAM D. MACMILLAN has received his discharge from the Ordnance Department and has resumed his duties as associate professor of astronomy at the University of Chicago.

Professor W. R. LONGLEY, of Yale University, has been on leave of absence, engaged in the ballistic division of the Dupont Company in Delaware.

Professor M. J. GOLDEN, emeritus professor of practical mechanics in Purdue University, and, since 1884, a member of the faculty, died on December 18, aged fifty-eight years.

To the list (published in this MONTHLY last September) of those who received the Doctorate in Mathematics in American universities, should be added the name of C. P. PAINE, Wisconsin. The title of his thesis was: "Modes of air motion and the equations of the general circulation of the earth's atmosphere."

Professor E. J. OGLESBY, after having served as instructor in gunnery, orientation and matériel at the Coast Artillery Training Camp, Fort Monroe, Va., with the rank of captain, resumed on January first the chair of mathematics at William and Mary College.

New instructors in mathematics at Northwestern University this year are Miss JESSICA M. YOUNG, Mr. T. DOLL, Dr. M. G. SMITH, and Mr. P. E. HEMKE. During the fall term five new assistants were also teaching mathematics. During last year Professor C. H. YEATON, Dr. C. E. WILDER, Mr. A. D. CAMPBELL and Mr. I. ROMAN left the department for military service.

At the University of Illinois, Assistant Professor J. E. MCATEE, of the mathematics department, died of pneumonia, following an attack of influenza, on December 1. His place has been filled by the appointment of Dr. F. W. REED, formerly connected with the astronomy department, and more recently with the instruction in the ground aviation school at the University. Mr. JOSEPH ROSENBACH, assistant and graduate student in mathematics, is just recovering from a severe attack of pneumonia and will be absent on sick leave until April 1.

Professor WARREN WEAVER, of Throop College of Technology, has been promoted to second lieutenant in the air service; he was recently engaged in experimental research work in the Science and Research Division.

From *Science* we learn that the departments of descriptive geometry and mechanical drawing and of mechanism and machine design at Stevens Institute of Technology have been combined to form a new department of machine design, of which FRANKLIN DER. FURMAN is professor and head. LEWIS E. ARMSTRONG, instructor at the Institute has been promoted to be an assistant professor of mathematics.

At the University of Chicago, the Students Army Training Corps, together with the Naval Unit and the Naval Auxiliary special students from the Municipal pier, numbered about two thousand. The surveying classes included about 350 students and there were about 600 students in navigation and trigonometry classes.

During November and December Professor J. W. YOUNG was on leave of absence from Dartmouth College, while serving as director of instruction in mathematics for the National War Work Council of the Young Men's Christian Associations of the United States. Professor F. M. MORGAN administered the affairs of the department of mathematics during his absence.

Professor PAUL HEEGAARD, of the University of Copenhagen has been appointed professor of mathematics at the University of Christiania on the understanding that he should lecture on geometry. His inaugural address, delivered in January, 1918, was entitled "Topics in the history of geometry."

Last autumn Professors E. H. MOORE, W. B. FITE, R. C. ARCHIBALD, H. G. KEPPEL and W. A. MANNING were appointed by the National War Work Council of the Young Men's Christian Association as a national committee to make a survey of the mathematical instruction given under the auspices of the Y. M. C. A. at the various Naval Stations, representing respectively the Central, Eastern, Northeastern, Southeastern and Western Departments. The object of this committee, working under the chairmanship of Professor G. A. MILLER, was to report on mathematical instruction given at these stations and to make suggestions tending to improving the instruction. As one result of reports thus made, a series of text-books in elementary mathematics is being prepared.

Commissions as second lieutenants were given to the following university teachers of mathematics, at Camp Zachary Taylor in November: Mr. R. W. BARNARD and Professor T. H. HILDEBRANDT, Michigan; Dr. L. R. FORD, Harvard; Mr. J. D. ESHLEMAN, Rochester; Dr. T. R. HOLLCROFT, Columbia; Dr. C. E. WILDER, Northwestern; Mr. H. E. WOLFE, Indiana.

Mr. Frank E. WOOD, of the meteorological section, Division of Science and Research, Signal Corps, and former professor of mathematics at the University of New Mexico, has been in charge of a group of computers engaged in the study of meteorological questions in the theory of exterior ballistics at the Aberdeen Proving Ground.

Two meetings of an Inter-Allied Scientific Conference have been held to consider international coöperation in science. The first meeting was in London under the auspices of the Royal Society, the second in Paris under the auspices of the Academy of Sciences of the Institute of France. The American delegates were: Dr. H. A. BUMSTEAD, Colonel J. J. CARTY, Professor W. F. DURAND, Dr. SIMON FLEXNER, Dr. GEORGE E. HALE, and Professor A. A. NOYES.

At Brown University the following appointments were made for the year 1918–19: Dr. A. B. FRIZELL as lecturer in mathematics, and Mr. T. A. CORNELL and Mr. W. R. BURWELL as instructors in mathematics. Mr. C. R. ADAMS, who was also an instructor for the first term, has been awarded the Grand Army Fellowship and is continuing graduate work in mathematics.

Professor K. O. E. LAMPE, of the Technische Hochschule, Berlin, died September 4, 1918, in his seventy-eighth year. He was appointed to the Hochschule staff in 1889 after twenty-four years of experience as teacher in Berlin secondary schools. He was an editor of *Jahrbuch über die Fortschritte der Mathematik* since 1885, and of *Archiv der Mathematik und Physik*, since 1900.

The list of mathematicians who spent the autumn working in coöperation with Captain VEBLEN at the Aberdeen Proving Grounds, on problems in ballistics, includes the following: Professor H. F. BLICHFELDT, Stanford University; Professor G. A. BLISS, University of Chicago; Professor W. C. GRAUSTEIN, Rice Institute; Dr. T. H. GRONWALL; Professors C. N. HASKINS and C. R. DINES, Dartmouth College; Professor H. H. MITCHELL, University of Pennsylvania; and Professor W. H. ROEVER, Washington University. Professors BLISS, HASKINS, and ROEVER have returned to their university work.

On account of conditions arising out of the war there will be no meeting of the American Mathematical Society in February this year.

This year for the first time in over twenty years Professor F. N. COLE, of Columbia University, secretary of the American Mathematical Society, took a vacation at Christmas time and did not perform the duties of secretary at the annual meeting of the Society.

At the meeting of the American Association of University Professors, held in Baltimore, December 27 and 28, Dr. A. O. LOVEJOY, professor of philosophy in the Johns Hopkins University, was elected president.

H. Y. BENEDICT, professor of mathematics, and dean of the College of Arts and Sciences at the University of Texas since 1911, was the compiler of the eight hundred and fifty page *Source Book relating to the History of the University of Texas: legislative, legal, bibliographical, statistical*, published in 1918 by the university, as Bulletin 1757.

Captain A. A. BENNETT, of the University of Texas, First Lieutenant T. BUCK, of the University of California, and First Lieutenant W. E. MILNE, of Bowdoin College, have finished their work with Major F. R. MOULTON in the Ordnance Department at Washington, D. C., and have again taken up academic occupations. Major Moulton expects to return to the University of Chicago in April.

At the regular meeting of the American Mathematical Society held at Chicago on Friday and Saturday, December 27 and 28, 1918, about eighty persons attended, including sixty members. On Friday afternoon President L. E. DICKSON delivered his retiring address on "Mathematics in War Perspective." Following this address there was a joint session with The Mathematical Association of America, which is to be reported fully in the next issue of this MONTHLY. During the Saturday sessions twenty-six papers were presented, the titles and abstracts of which will be found in the *Bulletin of the American Mathematical Society*. At a business meeting there was an election of officers, resulting as follows: President, Professor FRANK MORLEY; Vice-Presidents, Professors G. D. BIRKHOFF and FLORIAN CAJORI; Secretary, Professor F. N. COLE; Treasurer, Professor J. H. TANNER; Librarian, Professor D. E. SMITH; Committee of Publication, Professors F. N. COLE, VIRGIL SNYDER and J. W. YOUNG; Members of the Council, Professors H. E. HAWKES, W. A. HURWITZ, A. C. LUNN, and C. N. MOORE. At a meeting of the Council of the Society it was decided to postpone the Colloquium proposed for 1919 at Chicago until 1920. Further, it was voted that the incoming president be requested to appoint a committee of three to report to the Council at the April meeting on "steps which should be taken to organize or promote the publication, in America or elsewhere, of an adequate and comprehensive survey of the current mathematical literature of the world." The following committee was later appointed: Professors E. W. BROWN, G. A. MILLER, and R. C. ARCHIBALD (chairman).

P. PIAZETTI, professor of geodesy and celestial mechanics at the University of Pisa, died April 14, 1918, aged 58 years.

T. BONNESEN has been appointed professor of descriptive geometry at the Ecole Polytechnique, Copenhagen.

S. I. LATTÈS, for some time maître de conference at the University of Montpellier, but latterly a member of the faculty of sciences of the University of Toulouse, died July 5, 1918, aged 45 years. He received his doctorate from the University of Paris in 1906, and his thesis, *Sur les équations fonctionnelles qui definissent une courbe ou une surface invariante par une transformation*, occupies 137 pages of *Annali di matematica*, volume 13, series 3.

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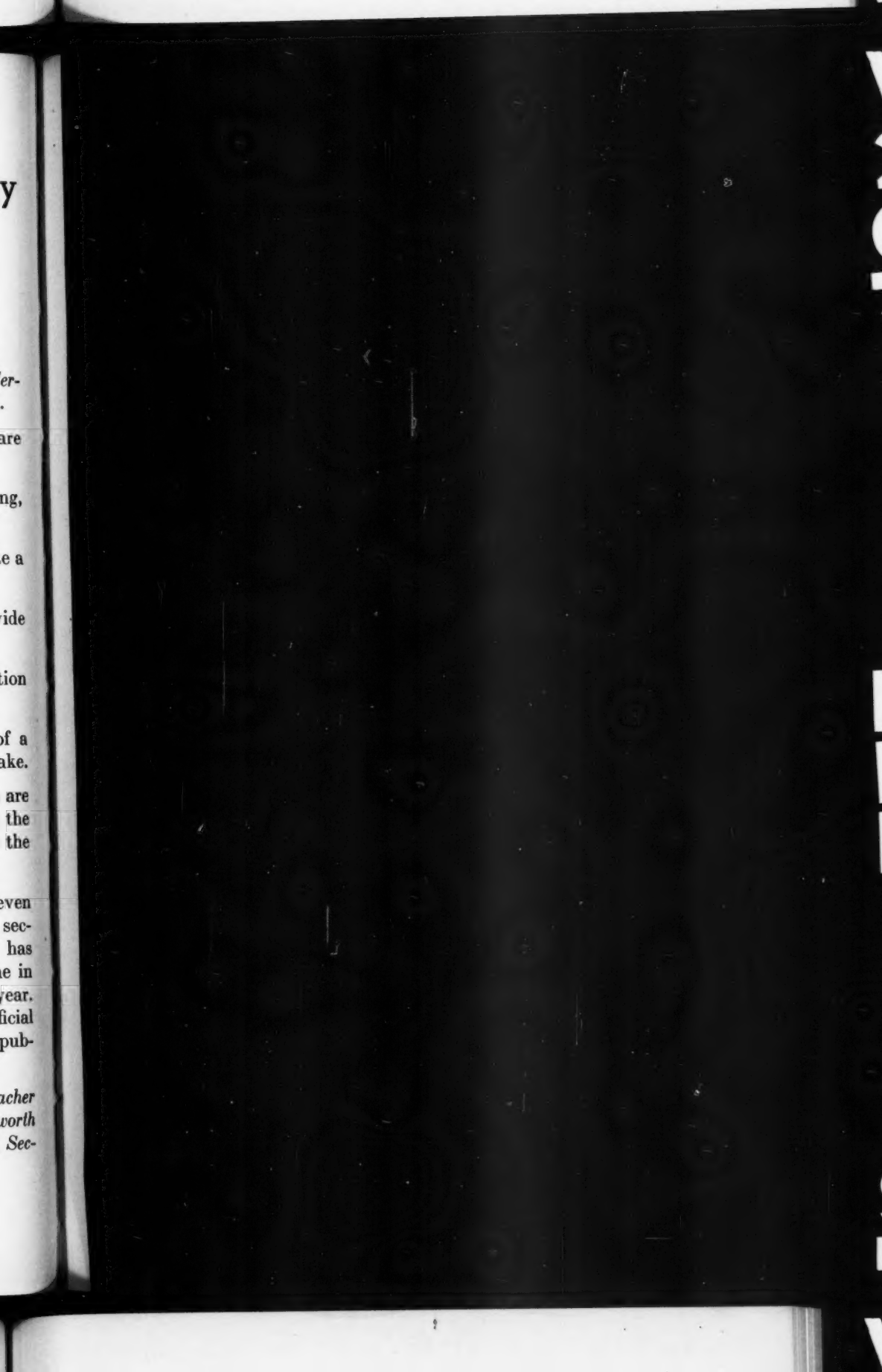
There are other journals suited to the Secondary field, and there are still others of technical scientific character in the University field: but the MONTHLY is the only journal of Collegiate grade in America suited to the needs of the non-specialist in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

*The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, Ohio.*



# THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

# THE MATHEMATICAL ASSOCIATION OF AMERICA

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## THIRD ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The third annual meeting[1] of the Association was held at the University of Chicago on Friday, December 27, 1918, in conjunction with the annual meeting of the American Mathematical Society, which was held on Friday and Saturday of that week. There were 86 in attendance at the sessions, including the following 73 members of the Association:

MARY ANDERSON, Illinois Woman's College.
R. C. ARCHIBALD, Brown University.
C. S. ATCHISON, Washington and Jefferson College.

R. P. BAKER, University of Iowa.
I. A. BARNETT, Washington University.
A. A. BENNETT, Aberdeen Proving Ground.
H. F. BLICHFELDT, Aberdeen Proving Ground.
HENRY BLUMBERG, University of Illinois.
J. W. BRADSHAW, University of Michigan.
H. T. BURGESS, University of Wisconsin.

W. D. CAIRNS, Oberlin College.
J. A. CAPARÓ, University of Notre Dame.
R. D. CARMICHAEL, University of Illinois.
C. C. CARTER, Chapin, Ill.
E. H. CARUS, La Salle, Ill.
H. E. COBB, Lewis Institute.
A. B. COBLE, University of Illinois.
C. E. COMSTOCK, Bradley Polytechnic Institute.
D. R. CURTISS, Northwestern University.

L. E. DICKSON, University of Chicago.
E. L. DODD, University of Texas.

E. B. ESCOTT, Peoples Life Insurance Company, Chicago, Ill.

W. J. FERGUSON, Ginn and Co., Chicago, Ill.
J. A. FOBERG, Crane Junior College, Chicago, Ill.
W. B. FORD, University of Michigan.
TOMLINSON FORT, University of Alabama.
A. F. FRUMVELLER, Marquette University.

O. E. GLENN, University of Pennsylvania.
ALICE BACHE GOULD, University of Chicago.
CORNELIUS GOUWENS, Graduate School, University of Chicago.

J. O. HASSLER, Crane Junior College, Chicago, Ill.
C. T. HAZARD, Purdue University.
OLIVE C. HAZLETT, Mount Holyoke College.
P. E. HEMKE, Northwestern University.
G. W. HESS, Shurtleff College.
T. H. HILDEBRANDT (University of Michigan), F. A. C. O. T. S. Instructor, Camp Taylor.

L. C. KARPINSKI, University of Michigan.
A. M. KENYON, Purdue University.
J. M. KINNEY, Hyde Park High School, Chicago, Ill.

KURT LAVES, University of Chicago.
SOLOMON LEFSCHETZ, University of Kansas.
A. C. LUNN, University of Chicago.

[1] The Association was founded in December, 1915.